Roma Tre University

PHD PROGRAMME IN MATHEMATICS
September 2021

Locating Ruelle-Pollicott resonances

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Abstract

We study the spectrum of transfer operators associated to various dynamical systems. Our aim is to obtain precise information on the discrete spectrum. To this end we propose a unitary approach. We consider various settings where new information can be obtained following different branches along the proposed path. These settings include affine expanding Markov maps, uniformly expanding Markov maps, non-uniformly expanding or simply monotone maps, hyperbolic diffeomorphisms. We believe this approach could be greatly generalized.

Acknowledgements

First, I would like to thank my supervisors, Prof. Carlangelo Liverani and Prof. Oliver Butterley, for their invaluable help, encouragement, and all they have taught me. Their insightful remarks, enthusiasm and broad interests in mathematics are always inspiring.

I am grateful to the faculty and graduate students at the Roma Tre Mathematics Departement for providing a pleasant working environment.

I would also like to thank my parents and family for their unconditional love and emotional support all these years.

Contents

Chapter 1

Introduction

Transfer operators are used widely in Dynamical Systems. Their first manifestations going back, at least, to the Koopman operator, and its use by John von Neumann to prove the mean ergodic theorem. Next, the Russian school developed the spectral theory for the Koopman operator acting on L^2 and its relation to the statistical properties of the system (such as ergodicity, mixing, ...), [27]. Later attention concentrated on the adjoint of the Koopman operator, now called the Frobenius-Perron or the Ruelle-Frobenius-Perron transfer operator. First such an operator appeared after coding the system [12]. Subsequently, starting with [57, 74], the direct study of the transfer operator acting on functions and, more recently, starting with [11], acting on spaces of distributions, acquired progressively more importance.[1] This is the current focus.

Historically research was mostly focussed on the study of the peripheral spectrum (which encodes sharp, quantitative, information on ergodicity and mixing), on establishing a spectral gap (which yields the rate of mixing) (e.g, [60, 6]), and on estimates of the essential spectrum and relations with the Ruelle zeta function (which encodes information on the spectrum of periodic orbits), see [7] for a recent review. For flows or systems with a neutral direction the study is often more involved but there is still the possibility of some type of spectral gap, e.g., [29, 16, 19, 17, 80, 23].

However, recently it has become apparent the need of a much deeper and detailed understanding of the point spectrum [39, 32, 33, 49, 47, 36, 18, 56, 48]. Possibilities include identifying the point spectrum by understanding the connection to the action of the dynamics on cohomology [36] or obtaining results related to bands of spectrum for transfer operators associated to systems with a neutral direction [35, 37, 38, 39, 18]. Additionally various works investigated the possibility of an explicit description of the spectrum for analytic expanding or hyperbolic maps [77, 9, 78] (using Blaschke products), or perturbative and generic results [55, 67, 1, 10]. Clearly, a more explicit description of the spectrum is important also in applications as it provides precise quantitative information on the invariant measures, entropy, decay of correlation, variance in the CLT and so on.

Unfortunately, no general theory exists to address this issue. One exception being the Hilbert metric technique, see [60], however such an approach can yield results only for the spectral gap and they are often far from optimal (see Remarks 4.2, 4.7 and 5.8), hence the need for an alternative approach. The special cases in which some results have been obtained seem to point to a general philosophy: to study the commutator between some type of differentiation and the transfer operator (e.g. see [30, 49, 36]). Although this idea is rather vague, we believe it can give rise to a general theory. In fact, it is surprising that this approach has not been explored in any systematic way, in spite of the vast literature devoted

[1] But see [76] for precursors of this point of view.

to transfer operators. Hence, the first step to substantiate our claim is an investigation of several concrete examples. This is the task of the present work [20].

We apply the above philosophy to different classes of dynamical system, starting from very simple ones and increasing progressively in complexity. For each example we obtain non trivial results that illustrate the power of this approach to the problem. Although our results fall short of a general theory, we believe they suffice to argue that walking further along this path is likely to yield interesting results in a much more general setting. Let us describe our results more in detail.

In section 3 we discuss the point spectrum for a family of transfer operators of Markov piecewise affine expanding maps. This is the simplest possible, non trivial, example. Yet, it goes a long way in illustrating our strategy.

In section 4 we address similar questions in the case of full branch piecewise smooth expanding maps of the interval. The simplest non-linear case. This is a class of maps which has been extensively studied and for which one could expect that all has been said already. Yet, we are able to obtain new interesting information. In particular, we concentrate on two transfer operators. The one associated to the SRB measure for which we obtain effective bounds on the spectral gap and fine informations about the spectra. The other is the operator associated to the measure of maximal entropy for which we establish a spectral gap of size at least $e^{h_{\rm top}} - 1$. This illustrates the fact, seen also later in other examples, that the transfer operator associated to the measure of maximal entropy enjoys surprisingly large gaps.

In section 5 we study the spectral gap for the operator associated to the measure of maximal entropy for full branch monotone maps. This includes the case of maps with attracting periodic orbits. We show that the measures of maximal entropy are exponentially mixing with a rate, at least, $h_{\rm top}$. We are not aware of similar results. Apart from the case of intermittent maps (when only neutral fixed points are present) for which it is known to exist a unique measure of maximal entropy which is exponentially mixing. However, even in this special case, nothing quantitatively precise was known on the speed of mixing.

Finally, in section 6, we study hyperbolic maps. We start, as an illustration, with automorphisms of the torus. This sheds some light on the difficulties involved in extending the approach to the general hyperbolic case. Next, we propose a possible solution to such difficulties: to study the spectrum of the action of the pushforward, for hyperbolic maps, on forms. This allows, for example, to study, again, the measure of maximal entropy. Once more we obtain a large gap. In particular, our approach provides alternative proofs, and a slight strengthening, of recent results by Baladi [8, Theorem 2.1] and Forni [40], moreover we establish a topological interpretation of the point spectrum which should hold in more generality.

Chapter 2

Background Material

2.1 Some Reminders of Functional Analysis

2.1.1 Bochner Integral

The Bochner integral extends the definition of Lebesgue integral to functions that take values in a Banach space. Let (M, Σ, μ) be a measure space and B a Banach space. First, a simple function is any finite sum of the form

$$s(x) = \sum_{i=1}^{n} \chi_{E_i}(x) b_i$$

where the E_i's are disjoint members of the σ-algebra Σ, the b_i's are distinct elements of B, and χ_E is the characteristic function of E. If $\mu(E_i)$ is finite whenever $b_i \neq 0$, then the simple function is integrable, and the integral is then defined by

$$\int_M \left[\sum_{i=1}^{n} \chi_{E_i}(x) b_i \right] d\mu = \sum_{i=1}^{n} \mu(E_i) b_i$$

exactly as it is for the ordinary Lebesgue integral. A measurable function $f : M \to B$ is Bochner integrable if there exists a sequence of integrable simple functions s_n such that

$$\lim_{n \to \infty} \int_M \|f - s_n\| \, d\mu = 0,$$

where the integral on the left-hand side is an ordinary Lebesgue integral. Here, the Banach space B is equipped with the Borel σ-algebra and $\| \cdot \|$ is continuous, so it is measurable. In this case, the Bochner integral is defined by

$$\int_M f \, d\mu = \lim_{n \to \infty} \int_M s_n \, d\mu.$$

Suppose there are sequences of integrable simple functions $\{s_n\}$ and $\{\tilde{s}_n\}$ such that

$$\lim_{n \to \infty} \int_M \|f - s_n\| \, d\mu = 0,$$
$$\lim_{n \to \infty} \int_M \|f - \tilde{s}_n\| \, d\mu = 0.$$

Then for $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for any $n \geq N$

$$\left\| \int_M s_n - \tilde{s}_n \, d\mu \right\| = \left\| \int_M s_n - f + f - \tilde{s}_n \, d\mu \right\| \leq \left\| \int_M s_n - f \right\| + \left\| \int_M f - \tilde{s}_n \, d\mu \right\| < \varepsilon.$$

So $\lim_{n\to\infty}\int_M s_n\, d\mu = \lim_{n\to\infty}\int_M \tilde{s}_n\, d\mu = \int_M f\, d\mu$. Therefore, $\int_M f\, d\mu$ does not depend on the sequence $\{s_n\}$.

2.1.2 Spectrum and Projections

Let T be a bounded linear operator acting on a Banach space X over the complex scalar field $\mathbb{C}$, and I be the identity operator on X. The spectrum of T is the set of all $\lambda \in \mathbb{C}$ for which the operator $\lambda - T$ does not have a bounded inverse.
The spectrum of a given operator T is often denoted by $\sigma(T)$, and its complement, denoted by $\rho(T) = \mathbb{C}\backslash\sigma(T)$, is called the resolvent set. The operator-valued function $\mathcal{R}(\lambda, T) = (\lambda - T)^{-1}$, defined on $\rho(T)$, is called the resolvent of T.
An eigenvalue of T is defined as a complex number λ such that there exists a nonzero vector $u \in X$, called an eigenvector, such that $Tu = \lambda u$. In other words λ is an eigenvalue if $N(\lambda - T)$ is not $\{0\}$, where $N(A)$ denotes the null space of a linear operator A.
A projector in a Banach space is a continuous linear operator P of the space into itself such that $P^2 = P$.

Lemma 2.1. *Let P be a projector in a Banach space X, then $R(P)$ is closed and $X = N(P) \oplus R(P)$, where $R(P)$ denotes the range of P.*

Proof. Since P is a projection, we have $P(x - Px) = 0$, for any $x \in X$. Hence $x - Px = y$, for some $y \in N(P)$. Thus $x = Px + y$. This shows that $X = R(P) + N(P)$. Now take $x \in R(P) \cap N(P)$. Since $x \in R(P)$, we have $x = Py$, for some $y \in X$. Applying P to both sides we get $Px = P^2y$. But $x \in N(P)$, hence $0 = Px = P^2y = Py = x$. This shows that $R(P) \cap N(P) = 0$ and so we have $X = R(P) \oplus N(P)$.
If $x \in R(P)$ then $x = Py$, for some $y \in X$. So $(1 - P)x = 0$ and $R(P) \subseteq N(1 - P)$. For $x \in N(1 - P)$, we have $x = Px$. So $x \in R(P)$ which implies $N(1 - P) \subseteq R(P)$. Therefore $R(P) = N(1 - P)$, which is closed. □

Theorem 2.2. *Let T be a bounded operator on a Banach space X. Let $\sigma(T) = S_1 \cup S_2$ be a decomposition of the spectrum of T such that there is a positively oriented simple closed curve γ within the resolvent set $\rho(T)$ which encloses an open set G_γ containing S_1 as its interior and S_2 as its exterior. Let*

$$P_\gamma = \frac{1}{2\pi i}\oint_\gamma \mathcal{R}(z, T)dz.$$

Then the operator P_γ is a projection and $P_\gamma T = TP_\gamma$. The spectrum of $T|_{R(P)}$ is S_1 and the spectrum of $T|_{N(P)}$ is S_2. As we already mentioned, the operator $\mathcal{R}(\lambda, T)$ takes value in the Banach space of linear bounded operators on X and the integration in the above formula is done in the sense of Bochner integral.

Proof. Since γ lies in the resolvent set, $\mathcal{R}(\lambda, T)$ is bounded on γ. It follows that $\|P_\gamma\| < \infty$. We show that $P_\gamma^2 = P_\gamma$.
Since the resolvent set is open, we can expand γ to a larger curve γ^* which contains $G_\gamma \cup \gamma$ in its interior and lies entirely within the resolvent set. Therefore by Cauchy's Theorem,

$$P_\gamma = \frac{1}{2\pi i}\oint_{\gamma^*} \mathcal{R}(z, T)dz.$$

Then

$$P_\gamma^2 = \frac{1}{(2\pi i)^2}\oint_\gamma \mathcal{R}(z, T)\oint_{\gamma^*} \mathcal{R}(\omega, T)d\omega dz.$$

Applying the identity

$$U^{-1} - V^{-1} = U^{-1}(V - U)V^{-1}$$

to $U = z - T$ and $V = \omega - T$, we can write

$$\mathcal{R}(z,T)\mathcal{R}(\omega,T) = \frac{\mathcal{R}(z,T) - \mathcal{R}(\omega,T)}{\omega - z}.$$

Therefore,

$$\begin{aligned} P_\gamma^2 &= \frac{1}{(2\pi i)^2} \oint_\gamma \oint_{\gamma^*} \frac{\mathcal{R}(z,T) - \mathcal{R}(\omega,T)}{\omega - z} d\omega dz \\ &= \frac{1}{(2\pi i)^2} \oint_\gamma \mathcal{R}(z,T)dz \oint_{\gamma^*} \frac{d\omega}{\omega - z} \\ &- \frac{1}{(2\pi i)^2} \oint_{\gamma^*} \mathcal{R}(\omega,T)d\omega \oint_\gamma \frac{dz}{\omega - z}. \end{aligned}$$

The first inner integral equals to $2\pi i$ by Cauchy's formula and the second inner integral is zero because ω is outside γ. So

$$P_\gamma^2 = \frac{1}{2\pi i} \oint_\gamma \mathcal{R}(z,T)dz = P_\gamma.$$

Since the resolvent of T commutes with T, it follows $P_\gamma T = TP_\gamma$.

We have already proved that T commutes with P_γ. So $TR(P_\gamma) \subset R(P_\gamma)$ and $TN(P_\gamma) \subset N(P_\gamma)$. Let $T_1 : R(P_\gamma) \to R(P_\gamma)$ be the restriction of T on $R(P_\gamma)$ and T_2 the restriction of T on $N(P_\gamma)$. We prove that $\lambda - T_1$ is invertible for $\lambda \in S_2$ which imples $\sigma(T_1) \subset S_1$. Actually we can construct its inverse explicitly as a contour integral:

$$B = \frac{1}{2\pi i} \oint_\gamma \frac{\mathcal{R}(z,T)}{\lambda - z} dz.$$

where γ is as in the statement of the theorem. In fact, since B commutes with T, it also commutes with P_γ. Hence

$$\begin{aligned} (\lambda - T)B &= \frac{1}{2\pi i} \oint_\gamma \frac{(\lambda - z)I + (zI - T)}{\lambda - z} \mathcal{R}(z,T)dz \\ &= \frac{1}{2\pi i} \oint_\gamma \Big(\mathcal{R}(z,T) + \frac{I}{\lambda - z}\Big) dz \\ &= P_\gamma + \frac{1}{2\pi i} \oint_\gamma \frac{I}{\lambda - z} dz \\ &= P_\gamma. \end{aligned}$$

Thus if $P_\gamma x = x$, $(\lambda I - T)Bx = x$. We also have $B(\lambda I - T)x = x$ because T and B commute. Thus the restriction of B on $R(P_\gamma)$ is the inverse of the restriction on $R(P_\gamma)$ of $\lambda - T$. Interchanging the roles of S_1 and S_2, we get $\sigma(T_2) \subset S_2$. □

Essential Spectrum

Our aim is to divide the spectrum $\sigma(T)$ into two parts $\sigma_d(T)$ and $\sigma_{ess}(T)$. The discrete spectrum of T, $\sigma_d(T)$, which consists of isolated points $\lambda \in \sigma(T)$ such that their associated Riesz projector has finite rank and the range of $\lambda - T$ is closed, and $\sigma_{ess}(T)$, the essential spectrum of T, which is going to be the remaining part of the spectrum. This motivates the following definition of the essential spectrum.

Definition 2.3. *[15] Let T be a bounded linear operator on a Banach space X. The (Browder) essential spectrum of T, $\sigma_{ess}(T)$, is the set of $\lambda \in \sigma(T)$, such that at least one of the following conditions holds:*

1) *The range of $\lambda - T$, $R(\lambda - T)$, is not closed;*

2) *$\bigcup_{r\geq 1} N(\lambda - T)^r$ is infinite dimensional;*

3) *λ is a limit point of $\sigma(T) \setminus \{\lambda\}$.*

There are many other definitions of the essential spectrum. For example, Wolf's [81] essential spectrum is the set of those $z \in \mathbb{C}$ such that $z - T$ is not Fredholm. Recall that an operator $T : X \to X$ is Fredholm if $R(T)$ is closed and dimensions of both $N(T)$ and the quotient $X \diagup R(T)$ are finite. The radius of the essential spectrum of T is the same for all different definitions [34], see [34, Section 1.4] and subsequent discussion.

2.1.3 Subspaces

Definition 2.4. *Let $V \subset X$ be a subspace of a normed vector space X. Given $x \in X$, we define the distance to V by:*

$$dist(x, V) = \inf\{\|x - y\| : y \in V\}.$$

Definition 2.5. *A subspace V is called a proper subspace of X if it is neither the whole space X nor the zero subspace* 0*.*

Lemma 2.6. *Let X be a Banach space, $V \subset X$ a proper closed subspace. Then for every $\varepsilon > 0$ there exists $x_0 \in X$ with $\|x_0\| = 1$ and $dist(x_0, V) \geq 1 - \varepsilon$.*

Proof. Let $x' \in X \setminus V$, then $d = dist(x', V) > 0$, (since V is closed). For each $\eta > 0$ there exists $y' \in V$ so that $d \leq \|x' - y'\| \leq d + \eta$. Let $x_0 = \frac{x'-y'}{\|x'-y'\|}$ and $\eta = \frac{\varepsilon d}{1-\varepsilon}$, for any $x \in V$ we have:

$$\|x_0 - x\| = \frac{1}{\|x' - y'\|}\|x' - y' - \|x' - y'\|x\| \geq \frac{d}{\|x' - y'\|} \geq \frac{d}{d + \eta} = 1 - \varepsilon,$$

since $y' + \|x' - y'\|x \in V$. □

Definition 2.7. *A normed vector space X is locally compact if any bounded sequence in X has a convergent subsequence.*

Theorem 2.8. *(S. Banach) Every locally compact Banach space X has finite dimension.*

Proof. Given linearly independent vectors $x_1, \cdots, x_r$ in X of unit norm, let $G_r \subset E$ be the r-dimensional subspace of X spanned by these vectors. Being finite-dimensional, G_r is a closed subspace of X. If it is a proper subspace, by the Lemma 2.6 we may find a unit vector $x_{r+1} \in X$ such that $\|x_{r+1} - x_i\| \geq \frac{1}{2}, i = 1, \cdots, r$.
If we may do this for each r, we obtain an infinite sequence $(x_r)_{r\geq 1}$ of unit vectors satisfying $\|x_p - x_q\| \geq \frac{1}{2}$ for each $p \neq q$, in particular admitting no convegent subsequence. This contradicts the assumption that X is locally compact. □

Definition 2.9. *A continuous map $F : X \to Y$ between topological spaces is called proper if $F^{-1}(M)$ is compact whenever $M \subset Y$ is compact.*

Let $L(X, Y)$ be the space of bounded linear maps from X to Y.

Lemma 2.10. *Let X and Y be complex Banach spaces and $S \in L(X, Y)$. If S restricted to closed, bounded sets is proper then $N(S)$, the null space of S, is finite dimensional and $R(S)$, the range of S, is closed.*

Proof. Since S is proper, $N(S) = S^{-1}(0)$ is locally compact. By Theorem 2.8, $N(S)$ is finite dimensional.

Next we prove that $R(S)$ is closed. Let $\{x_n\}$ be a sequence in X such that $\{S(x_n)\}$ is a Cauchy sequence on Y. We need to show that $\{S(x_n)\}$ converges to a point $y \in R(S)$. Since Y is Banach, $\{S(x_n)\}$ is convergent. The set $\{S(x_n)\}$ with its limit is compact so by hypothesis $\{x_i\}$ has a convergent subsequence, let us call x the limit. Since T is continuous, $S(x) = y$. □

2.1.4 Measure of Noncompactness

Let X be a complete Banach space and A a bounded subset of X.

Definition 2.11. *We define $\gamma(A)$, which we call the measure of noncompactness of A, to be the infimum of $d > 0$ such that there exists a finite number of sets $S_1, \cdots, S_n$ with diameter(S_i)$\leq d$ and $A = \bigcup_{i=1}^{n} S_i$.*

Definition 2.12. *We call the ball measure of noncompactness of A in X, $\tilde{\gamma}_X(A)$, to be the infimum of $r > 0$ such that there exists a finite number of balls $V_1, \cdots, V_n$ with centers in X and radii r and $A \subset \bigcup_{i=1}^{n} V_i$.*

Definition 2.13. *If X_1 and X_2 are Banach spaces and $T \in L(X_1, X_2)$, we say that T is a k-set-contraction if for every bounded set $A \subset X_1$, $\gamma_{X_2}(T(A)) \leq k\gamma_{X_1}(A)$, and we say that T is a ball-k-set-contraction if $\tilde{\gamma}_{X_2}(T(A)) \leq k\tilde{\gamma}_{X_1}(A)$ for every bounded set A in X_1.*
We define $\gamma(T) = \inf\{k > 0 : T$ is a k-set-contraction$\}$ and $\tilde{\gamma}(T) = \inf\{k > 0 : T$ is a ball-k-set-contraction$\}$.

Remark 2.14. The above ideas can also be defined for nonlinear maps between metric spaces [28, 68, 69].

Denote the closed ideal of compact linear operators of X into X by K. Let $Z = L(X, X)/K$. We define a seminorm $\|T\|_K$ on $L(X, X)$ by $\|T\|_K = \inf_{C \in K} \|T + C\|$, and $\|T\|_K$ induces a norm on Z with respect to which Z is a complete normed space.

Lemma 2.15. *The measure of noncompactness and the ball measure of noncompactness satisfy the following properties:*

a) Let $A \subseteq X$, then $\bar{A}$ is compact if and only if $\tilde{\gamma}(A) = 0$. Also, $\bar{A}$ is compact if and only if $\gamma(A) = 0$.

b) An operator $T \in L(X, X)$ is compact if and only if $\tilde{\gamma}(T) = 0$. Also, T is compact if and only if $\gamma(T) = 0$.

c) $\gamma(T) \leq \|T\|$.

d) For bounded subsets $A, B \subseteq X$, we have $\gamma(A + B) \leq \gamma(A) + \gamma(B)$ and $\tilde{\gamma}(A + B) \leq \tilde{\gamma}(A) + \tilde{\gamma}(B)$.

Proof. a) For $\varepsilon > 0$, since $\bar{A}$ is compact, A can be covered by a finite number of balls of radius ε. Since ε is arbitrary, we have $\tilde{\gamma}(A) = 0$. Therefore $\gamma(A) = 0$, because $\gamma(A) \leq \tilde{\gamma}(A)$. Now assume that $\bar{A}$ is not compact, then there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \bar{A}$ which has no accumulation point in $\bar{A}$. Define $B_\varepsilon(x_n) := \{y \in X : \|x_n - y\| < \varepsilon\}$. Then there exists a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ such that for any $i, j \in \mathbb{N}$, $B_\varepsilon(x_{n_i}) \cap B_\varepsilon(x_{n_j}) = \emptyset$, for some $\varepsilon > 0$. If not, then for any $\varepsilon > 0$, there is $N \in \mathbb{N}$, such that for any $n, m \geq N$, $|x_n - x_m| < 2\varepsilon$. So $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence which is Cauchy and therefore it has an accumulation point in $\bar{A}$, which is in contrary to the assumption. So we conclude that $\tilde{\gamma}(A) \geq \gamma(A) > \varepsilon$.

b) First suppose that T is a compact operator. For any bounded set $A \subseteq X$, $\overline{T(A)}$ is compact. So by (a), $\tilde{\gamma}(T(A)) = 0$ and $\gamma(T(A)) = 0$. Hence for any $k > 0$, T is a ball-k-set-contraction and a k-set-contraction. So $\tilde{\gamma}(T) = 0$ and $\gamma(T) = 0$.
Now assume that $\gamma(T) = 0$. Let $A \subseteq X$, be a ball of radius $R > 0$. For $\varepsilon > 0$, we have $\gamma(T) < \frac{\varepsilon}{R}$. Therefore $\gamma(T(A)) < \frac{\varepsilon}{R}\gamma(A) < \varepsilon$. So $\gamma(T(A)) = 0$, then (a) implies $\overline{T(A)}$ is compact. So T is a compact operator. The same proof works for the case $\tilde{\gamma}(T) = 0$.
c) If $\gamma(A) = r$, then for $\lambda > r$, there is a covering of A by finitely many sets $\{B_i\}_{i=1}^n$ of diameter not greater than λ. So $\{T(B_i)\}_{i=1}^n$ will cover $T(A)$. For any $1 \leq i \leq n$

$$diam(T(B_i)) = \sup_{x,y\in B_i} \|Tx - Ty\| \leq \|T\| \sup_{x,y\in B_i} \|x - y\| \leq \|T\|\lambda,$$

which implies $\gamma(T) \leq \|T\|$.
d) Let $\gamma(A) = \alpha$ and $\gamma(B) = \beta$. Then for $r > \alpha$, there is a covering of A by a finite number of sets $\{a_i\}_{i=1}^n$ of diameter not greater than r and for $\rho > \beta$, there is a covering of B by a finite number of sets $\{b_j\}_{j=1}^m$ of diameter not greater than ρ. So $A + B = \{x + y\}_{x\in A, y\in B} \subseteq \cup_{i,j}\{x + y\}_{x\in a_i, y\in b_j}$. For any $1 \leq i \leq n, 1 \leq j \leq m$ and $x, x' \in a_i$, $y, y' \in b_j$ we have

$$\|x + y - x' - y'\| \leq \|x - x'\| + \|y - y'\| \leq r + \rho.$$

Therefore $\gamma(A + B) \leq \gamma(A) + \gamma(B)$.
Now let $\tilde{\gamma}(A) = \kappa$ and $\tilde{\gamma}(B) = \lambda$. Then for $\mu > \kappa$, there is a covering of A by a finite number of balls $\{B(a_i, r_i)\}_{i=1}^n$ of radius $r_i \leq \mu$ and for $\nu > \lambda$, there is a covering of B by a finite number of balls $\{B(b_j, \rho_j)\}_{j=1}^m$ of radius $\rho_j \leq \nu$. So $A + B = \{x + y\}_{x\in A, y\in B} \subseteq \cup_{i,j}\{x + y\}_{x\in B(a_i, r_i), y\in B(b_j, \rho_j)}$. For any $1 \leq i \leq n, 1 \leq j \leq m$ and $x \in B(a_i, r_i)$, $y \in B(b_j, \rho_j)$ we have

$$\|x + y - (a_i + b_j)\| \leq \|x - a_i\| + \|y - b_j\| \leq \mu + \nu.$$

Therefore $\tilde{\gamma}(A + B) \leq \tilde{\gamma}(A) + \tilde{\gamma}(B)$.

□

Lemma 2.16. *Let X and Y be complex Banach spaces and $T \in L(X, Y)$. Then we have $\gamma(T^*) \leq \tilde{\gamma}(T)$.*

Proof. Suppose T is a ball-k-set-contraction. To show that T^* is a k-set-contraction, it suffices to show that if S is a set of diameter less than or equal to d in Y^*, $T^*(S)$ can be covered by a finite number of sets of diameter less than or equal to $kd + \varepsilon$, for any $\varepsilon > 0$.
Consider $T(B)$, where $B = \{x \in X, ||x|| \leq 1\}$. Since $\tilde{\gamma}(B) \leq 1$ and T is a ball-k-set-contraction, $T(B)$ can be covered by a finite number of balls $B_{k+\frac{\varepsilon}{2d}}(y_i)$ in Y, $1 \leq i \leq n$, with centers at y_i, and radii $k + \frac{\varepsilon}{2d}$. Select M such that $\|y_i\| \leq M$, $1 \leq i \leq n$, and $\|y^*\| \leq M$ for all $y^* \in S$. Hence, we have $|y^*(y_i)| \leq M^2$ for each $y^* \in S$. Decompose the closed interval $[-M^2, M^2]$ into a union of disjoint intervals Δ_i, $1 \leq i \leq p$, of length less than $\frac{\varepsilon}{2}$. We consider an equivalence relation as follows: Given y_1^* and $y_2^* \in S$, write $y_1^* \sim y_2^*$ iff for each i, $1 \leq i \leq n$, $y_1^*(y_i)$ and $y_2^*(y_i)$ lie in the same interval $\Delta_{j(i)}$, $1 \leq j(i) \leq p$. Then we divide S

into equivalence classes S_j, $1 \le j \le q$,
We claim that diameter $(T^*(S_i)) \le kd + \varepsilon$. Take y_1^* and y_2^* in S_i. We have

$$\|T^*(y_1^*) - T^*(y_2^*)\| = \sup_{x \in B} |y_1^*(Tx) - y_2^*(Tx)| = \sup_{y \in T(B)} |y_1^*(y) - y_2^*(y)|.$$

If $y \in T(B)$, we know that $y \in B_{k+\frac{\varepsilon}{2}}(y_i)$ for some i, $1 \le i \le n$. It follows that

$$|y_1^*(y) - y_2^*(y)| \le |y_1^*(y - y_i) - y_2^*(y - y_i)| + |y_1^*(y_i) - y_2^*(y_i)|$$

$$= |(y_1^* - y_2^*)(y - y_i)| + |y_1^*(y_i) - y_2^*(y_i)| \le d(k + \frac{\varepsilon}{2d}) + \frac{\varepsilon}{2} = kd + \varepsilon.$$

Thus, for each $\varepsilon > 0$, $\|T^*(y_1^*) - T^*(y_2^*)\| \le kd + \varepsilon$. This shows that diameter $(T^*(S_i)) \le kd + \varepsilon$, and since $T^*(S) \subset \bigcup_{i=1}^{q} T^*(S_i)$, we have covered $T^*(S)$ by a finite number of sets of diameter less than or equal to $kd + \varepsilon$. □

Lemma 2.17. *Let X be a complex Banach space and $T \in L(X, X)$. Assume that for some $n \ge 1, \tilde{\gamma}(T^n) < 1$. Then for any $r \ge 1, (I - T)^r$ restricted to closed, bounded sets is proper.*

Proof. Let A be a closed, bounded set in X and M a compact set. We have to show that $M_1 = \{x \in A : (I - T)x \in M\}$ is compact. By Lemma 2.15, in order to show that M_1 is compact it suffices to show that $\tilde{\gamma}(M_1) = 0$. Notice that $\tilde{\gamma}(M_1)$ is defined, since A is bounded. Suppose $x \in M_1$, so that $x = Tx + m$ for some $m \in M$. Substituting for x on the right, $x = T^2x + Tm + m$, and continuing in this way we find

$$x = T^n x + \sum_{i=0}^{n-1} T^i m. \tag{2.1}$$

If we write $M_* = \sum_{i=0}^{n-1} T^i(M)$, M_* is compact, since it is the continuous image of a compact set. Furthermore, (2.1) implies that $M_1 \subset T^n(M_1) + M_*$, so that $\tilde{\gamma}(M_1) \le \tilde{\gamma}(T^n(M_1))$, by Lemma 2.15. Since T^n is a ball-k-set-contraction, $k < 1$, $\tilde{\gamma}(M_1) \le k\tilde{\gamma}(M_1)$. It follows that $\tilde{\gamma}(M_1) = 0$. Hence $1 - T$ is proper.
We want to show that $(1 - T)^r, r > 1$ is proper. We proceed by induction. Assume that for $r > 1$, $(1 - T)^{(r-1)}$ is proper, then for compact set M, $(1 - T)^{-(r-1)}(M)$ is compact. So $(1-T)^{-r}(M) = (1-T)^{-1}(1-T)^{-(r-1)}(M)$ is also compact. Therefore $(1-T)^r$ is proper. □

2.2 Nussbaum formula for essential spectral radius

In this section, we obtain a characterization of the essential spectral radius $r_e = \sup\{|\lambda| : \lambda \in \sigma_{ess}(T)\}$. We essentially follow [70].

Lemma 2.18. *Let X be a Banach space and $T \in L(X, X)$. Let $r_e' := \inf_n (\tilde{\gamma}(T^n))^{\frac{1}{n}}$. Then $\lim_{n\to\infty} (\tilde{\gamma}(T^n))^{\frac{1}{n}}$ and $\lim_{n\to\infty} (\gamma(T^n))^{\frac{1}{n}}$ exist and equal r_e', and if $|\lambda| > r_e'$, $N(\lambda - T)^r$ is finite dimensional for any $r \ge 1$ and $R(\lambda - T)$ is closed.*

Proof. We start showing that $\limsup_{n\to\infty} (\tilde{\gamma}(T^n))^{\frac{1}{n}} \le r_e'$.

For any $\varepsilon > 0$, choose m such that $(\tilde{\gamma}(T^m))^{\frac{1}{m}} \le r_e' + \varepsilon$. For large enough n, write $n = pm + q$ where $0 \le q \le (m - 1)$.

For all $S \in L(X, X)$, $A \subseteq X$, we have:

$$\tilde{\gamma}(S(A)) \le \tilde{\gamma}(S)\tilde{\gamma}(A)$$

Hence for all $S, T \in L(X, X)$, $A \subseteq X$

$$\tilde{\gamma}(ST(A)) \le \tilde{\gamma}(S)\tilde{\gamma}(T(A)) \le \tilde{\gamma}(S)\tilde{\gamma}(T)\tilde{\gamma}(A).$$

Therefore $\tilde{\gamma}$ has the submultiplicative property:

$$\tilde{\gamma}(ST) \le \tilde{\gamma}(S)\tilde{\gamma}(T).$$

Then, by the above fact and $\tilde{\gamma}(T) \ge 0$ for $T \in L(X, X)$, we obtain

$$(\tilde{\gamma}(T^n))^{\frac{1}{n}} \le (\tilde{\gamma}(T^m))^{\frac{p}{n}} \cdot (\tilde{\gamma}(T))^{\frac{q}{n}} \le (r'_e + \varepsilon)^{\frac{pm}{n}} (\tilde{\gamma}(T))^{\frac{q}{n}}.$$

Since $\frac{pm}{n} \to 1$ and $\frac{q}{n} \to 0$ as $n \to \infty$, we must have $\limsup_{n\to\infty}(\tilde{\gamma}(T^n))^{\frac{1}{n}} \le r'_e + \varepsilon$. Since ε was arbitrary, we have proved $\limsup_{n\to\infty}(\tilde{\gamma}(T^n))^{\frac{1}{n}} \le r'_e \le \liminf_{n\to\infty}(\tilde{\gamma}(T^n))^{\frac{1}{n}}$. Therefore $\lim_{n\to\infty}(\tilde{\gamma}(T^n))^{\frac{1}{n}}$ exists. In the exact same way, we can prove that $\lim_{n\to\infty}(\gamma(T^n))^{\frac{1}{n}}$ exists. Suppose $|\lambda| > r'_e$ and n such that $(\tilde{\gamma}(T^n))^{\frac{1}{n}} < |\lambda|$. Consider $T_1 = (\frac{1}{\lambda})T$ and notice that $\tilde{\gamma}(T_1^n) = (\frac{1}{|\lambda|})\tilde{\gamma}(T^n) = k < 1$. By Lemma 2.17, $(I - T_1)^r$, for any $r \ge 1$ is proper on closed, bounded sets. By Lemma 2.10, $N(I - T_1)^r$ is finite dimensional for any $r \ge 1$, $R(I - T_1)$ is closed. □

Lemma 2.19. *If $|\lambda_0| > r'_e$, then λ_0 is not a limit point of $\sigma(T) \setminus \{\lambda_0\}$.*

Proof. We show that all points $\lambda \ne \lambda_0$, in some neighborhood of the point λ_0, belong to the resolvent of T and so λ_0 is not a limit point of $\sigma(T)$. The case $\lambda_0 \in \rho(T)$ is tivial. Let $\lambda_0 \in \sigma(T)$. First we prove that either $N(\lambda_0 - T) \ne 0$ or $N(\lambda_0 - T^*) \ne 0$.
Suppose that $N(\lambda_0 - T) = N(\lambda_0 - T^*) = 0$. Then $(\lambda_0 - T)^{-1} : D \to X$ exists on $D = R(\lambda_0 - T)$ which is closed, by Lemma 2.18 applied to λ_0. Assume that $D \ne X$, then by Lemma 2.6, there is $u \in X$, such that $\|u\| = 1$ and $\|u - w\| \ge \frac{1}{2}$ for any $w \in D$. Let $V := span\{u, D\}$, then for any $v \in V$ we can write $v = \alpha u + w$ with $w \in D$. Define $l(v) := \alpha$, then

$$\|v\| = \|\alpha u + w\| = |\alpha|\|u - (-\alpha^{-1}w)\| \ge \frac{1}{2}|\alpha| = \frac{1}{2}|l(v)|.$$

So

$$|l(v)| \le 2\|v\|.$$

We can then apply the Hahn-Banach theorem, and we have an extension of l on all X and $l \ne 0$, since $l(u) = 1$. For any $v \in X$, $(\lambda_0 - T^*)l(v) = l\big((\lambda_0 - T)v\big) = 0$. So $(\lambda_0 - T^*)l = 0$. This contradicts $N(\lambda_0 - T^*) = 0$. So $D = X$, which implies that $\lambda_0 - T$ is invertible on X and by the bounded inverse theorem, $(\lambda_0 - T)^{-1}$ is a bounded operator. Therefore $\lambda_0 \notin \sigma(T)$ and this contradicts the assumption.
Suppose that there exists a sequence $\{\tilde{\lambda}_n\}_{n=1}^{\infty} \subset \sigma(T) \setminus \{\lambda_0\}$ which accumulates to λ_0.
Then there are either infinitely many $\tilde{u}_n \in N(\tilde{\lambda}_n - T)$ or infinitely many $\tilde{l}_n \in N(\tilde{\lambda}_n - T^*)$.
For each $\varepsilon > 0$, there exists $\bar{n} \in \mathbb{N}$ such that, for $n > \bar{n}$, $|\tilde{\lambda}_n - \lambda_0| < \varepsilon|\lambda_0|$.
In the first case, for any $k \in \mathbb{N}$, let M_k be the subspace spanned by the vectors $\tilde{u}_{\bar{n}}, \cdots, \tilde{u}_{\bar{n}+k}$. Set $u_k := \tilde{u}_{\bar{n}+k}$ and $\lambda_k := \tilde{\lambda}_{\bar{n}+k}$. Since $u_1, u_2, \cdots$ are linearly independent, each M_{k-1} is a closed proper subspace of M_k. So, by Lemma 2.6, there exists $v_k \in M_k$, such that $\|v_k\| = 1$ and $d(v_k, M_{k-1}) \ge 1 - \varepsilon$.
Note that $v_k = \alpha_k u_k + w_k$ where $\alpha_k \in \mathbb{R}, w_k \in M_{k-1}$.
So for $k, r, s \in \mathbb{N}$, such that $s > k$

$$\|T^r v_s - T^r v_k\| = \|T^r(\alpha_s u_s) + T^r w_s - T^r v_k\| = \|\alpha_s \lambda_s^r u_s + T^r w_s - T^r v_k\|$$

$$= |\lambda_s^r| \|v_s - (w_s - \lambda_s^{-r}T^r w_s + \lambda_s^{-r}T^r v_k)\| \geq |\lambda_s^r|(1-\varepsilon) = |(\lambda_s - \lambda_0 + \lambda_0)^r|(1-\varepsilon)$$
$$= |\lambda_0^r| \Big|1 + \frac{\lambda_s - \lambda_0}{\lambda_0}\Big|^r (1-\varepsilon) \geq |\lambda_0|^r \big(1 - \Big|\frac{\lambda_s - \lambda_0}{\lambda_0}\Big|\big)^r (1-\varepsilon) \geq |\lambda_0|^r (1-\varepsilon)^{r+1}.$$

This implies that $T^r\{|v| \leq 1\}$ cannot be covered with finitely many sets of diameter $\frac{1}{4}|\lambda_0|^r(1-\varepsilon)^{r+1}$. Therefore, by arbitrariness of ε, $\tilde{\gamma}(T^r) \geq \gamma(T^r) \geq \frac{1}{4}|\lambda_0|^r$.
In the second case, exactly the same argument implies $\gamma(T^{*r}) \geq \frac{1}{4}|\lambda_0|^r$. By Lemma 2.16, $\tilde{\gamma}(T^r) \geq \frac{1}{4}|\lambda_0|^r$.
Thus in both cases, $r'_e = \inf_n(\tilde{\gamma}(T^n))^{\frac{1}{n}} \geq |\lambda_0|$ which contradicts the assumption. So λ_0 is not a limit point of $\sigma(T)$. ☐

Corollary 2.20. *According to definition of the essential spectrum, Lemma 2.18 and Lemma 2.19 imply that* $r'_e \geq r_e$.

Lemma 2.21. *Let T be as above and $r_e = \sup\{|\lambda| : \lambda \in \sigma_{ess}(T)\}$. Take $r > r_e$. Then there exists a finite dimensional linear operator F such that $\sigma(T+F) \subset \{\lambda : |\lambda| \leq r\}$.*

Proof. Since $\sigma(T) \cap \{\lambda : |\lambda| \geq r\}$ is a compact set of isolated points, it consists of a finite number of points $\lambda_1, \cdots, \lambda_n$. Let C_i be a small circle about λ_i, $C_j \cap C_j = \emptyset$ for $i \neq j$ and containing only λ_i from $\sigma(T)$, and $P_i = (\frac{1}{2\pi i})\int_{C_i}(\lambda - T)^{-1}d\lambda$ be the Riesz projector associated to λ_i. Since λ_i does not belong to the essential spectrum, $R(P_i)$ which is the eigenspace associated to λ_i is finite dimensional. If we write $P = \sum_{i=1}^n P_i$, we therefore see that P is a finite dimensional projection. We take $F = TP$.
Let us write $N = N(P)$, the null space of P, and $R = R(P)$, the range of P, and note that $X = N \oplus R$. Consider $\lambda - T - F$ for $|\lambda| > r$. For $|\lambda| > r$ and $\lambda \neq \lambda_i$, $1 \leq i \leq n$, we have $\lambda \in \rho(T)$. Then it is clear that $(\lambda - T - F)|N = \lambda - T|N$ is a one to one map of N onto N. Furthermore $(\lambda - T - F)|R = \lambda|R$, which is clearly one to one and onto for $|\lambda| > r$. Thus $\lambda - T - F$ is a one to one map of X for $|\lambda| > r$. ☐

The following lemma is not necessary for our applications but we include it for completeness.

Lemma 2.22. *Let X be a complex Banach space and $T \in L(X,X)$. Then $\lim_{n\to\infty}(\gamma(T^n))^{\frac{1}{n}}$, $\lim_{n\to\infty}(\tilde{\gamma}(T^n))^{\frac{1}{n}}$ and $\lim_{n\to\infty}(\|T^n\|_K))^{\frac{1}{n}}$ are all equal to r_e.*

Proof. We have already seen in Lemma 2.18 that $\lim_{n\to\infty}(\tilde{\gamma}(T^n))^{\frac{1}{n}}$ and $\lim_{n\to\infty}(\gamma(T^n))^{\frac{1}{n}}$ exist and equal to r'_e. The same argument as in Lemma 2.18 shows that $r''_e := \lim_{n\to\infty}\|T^n\|_K^{\frac{1}{n}}$ exists. For $S \in L(X,X)$ and any compact operator $C \in L(X,X)$, $\gamma(S) = \gamma(S+C) \leq \|S+C\|$. Therefore $\gamma(S) \leq \|S\|_K$, which implies $r'_e \leq r''_e$.
Now we show that $r''_e \leq r_e$. Suppose not, so that $r_e < r''_e$, and select $r_e < r < r''_e$. For this r, let F be as in Lemma 2.21 and write $T_1 = T + F$. By the ordinary spectral radius theorem we know that $\lim_{n\longrightarrow\infty}\|T_1^n\|^{\frac{1}{n}} \leq r$. On the other hand, $\|T^n\|_K \leq \|T_1^n\|$, so that we obtain $r''_e = \lim_{n\longrightarrow\infty}\|T^n\|_K^{\frac{1}{n}} \leq r$, a contradiction. It follows that $r''_e \leq r_e$. Now by Corollary 2.20, we have $r_e = r'_e = r''_e$. ☐

2.2.1 The proof of Hennion theorem using Nussbaum Formula

Theorem 2.23. *(Hennion argument [50]). Consider two Banach spaces $X \subset X_w$, $\|\cdot\| \geq \|\cdot\|_w$, and an operator $T \in L(X_w, X_w)$ and its restriciton to X such that, for some $M > \theta > 0$, $A, B, C > 0$, and for each $n \in \mathbb{N}, f \in X$, holds true*

$$\|T^n f\|_w \leq CM^n\|f\|_w; \quad \|T^n f\| \leq A\theta^n\|f\| + BM^n\|f\|_w$$

Then the spectral radius of T is bounded by M. If, in addition, T is compact as an operator from X to X_w, then the essential spectral radius of T is bounded by θ.

Proof. Using the spectral radius formula and hypothesis of the theorem, we have

$$\rho(T) = \lim_{n\to\infty} \|T^n\|_w^{\frac{1}{n}} \le \lim_{n\to\infty} (CM^n)^{\frac{1}{n}} = M$$

which implies the first assertion.
For the second part, notice that by Lemma 2.22

$$r_e = \lim_{n\to\infty} \sqrt[n]{\tilde{\gamma}(T^n)} \le \lim_{n\to\infty} \sqrt[n]{\tilde{\gamma}(T^n B_1)}$$

where $B_1 := \{f \in X \mid \|f\| \le 1\}$.
Now we prove that $T^n B_1$ can be covered by a finite number of balls of radius $const \cdot \theta^n$, which implies that $r_e \le \lim_{n\to\infty} \sqrt[n]{\tilde{\gamma}(T^n B_1)} \le \lim_{n\to\infty} \sqrt[n]{const \cdot \theta^n} = \theta$. By hypotheses TB_1 is relatively compact in X_w. Thus, for each $\varepsilon > 0$ there are $f_1, \cdots, f_{N_\varepsilon} \in TB_1$ such that $TB_1 \subseteq \bigcup_{i=1}^{N_\varepsilon} U_\varepsilon(f_i)$, where $U_\varepsilon(f_i) = \{f \in X \mid \|f - f_i\|_w < \varepsilon\}$.
Since f, f_i belong to $T(B_1)$, there are $g, g_i \in B_1$ such that $f = T(g), f_i = T(g_i)$. So

$$\|f - f_i\| = \|T(g - g_i)\| \le A\theta\|g - g_i\| + BM\|g - g_i\| \le A\theta + BM.$$

For $f \in TB_1 \cap U_\varepsilon(f_i)$, holds

$$\|T^{n-1}(f - f_i)\| \le A\theta^{n-1}\|f - f_i\| + BM^{n-1}\|f - f_i\|_w \le A\theta^{n-1}(A\theta + BM) + BM^{n-1}\varepsilon.$$

Choosing ε sufficiently small we can conclude that for each $n \in \mathbb{N}$ the set $T^n(B_1)$ can be covered by a finite number of $\|\cdot\|$-balls of radius $const \cdot \theta^n$ centered at the points $\{T^{n-1} f_i\}_{i=1}^{N_\varepsilon}$. □

Chapter 3

Affine Expanding Markov maps

In this section we discuss the simplest possible case: one dimensional piecewise affine Markov maps. This not only allows us to show our approach in the simplest possible form but, amazingly, yields results that we have not been able to locate in the literature. In this setting the invariant densities can be computed easily since the Frobenius-Perron operator can be represented by a finite-dimensional matrix (see [13, Chapter 9] for full details).

Here we go beyond the peripheral spectrum and show that studying a particular family of matrices yields the full Ruelle-Pollicott spectrum. To this end, the smoothness of the observables is relevant. This will be a leitmotiv in the following and it is essential since it is known that even the point spectrum of the transfer operator may change drastically if one considers a class of observables that allow discontinuities (e.g., see [14, 11] and also Remark 3.3).

Let $I := [0, 1]$ and let $f : I \to I$ be a *piecewise affine expanding Markov map* in the following sense: there exists a collection of disjoint open intervals $\{I_j\}_{j=1}^N = \{(p_j, p_{j+1})\}_{j=1}^N$ which form a partition of a full measure subset of I and, for all i, j,

$$\text{either} \quad f(I_i) \cap I_j = \emptyset, \quad \text{or} \quad I_j \subseteq f(I_i).$$

Moreover, we suppose that f' is constant on each I_i. Finally we suppose that there exists $\lambda > 1$ such that[1] $f'(x) \geq \lambda$ for all $x \in \cup_i I_i$.

The partition $\{I_i\}_{i=1}^N$ is called a *Markov partition*. Let $\mathcal{I} = \cup_{i=1}^N I_i$ be the disjoint union of the partition elements. The $N \times N$ matrix A defined by

$$A[i, j] = 1, \text{ if } I_j \subseteq f(I_i), \text{ and } A[i, j] = 0, \text{ if } f(I_i) \cap I_j = \emptyset,$$

is called the *adjacency matrix*[2] of the Markov map f. For convenience let $\lambda_j := f'|_{I_j}$ and $\lambda = \min_j \lambda_j$. For any[3] $k \in \mathbb{N}_0$, let B_k be the $N \times N$ matrix defined by

$$B_k[i, j] := \lambda_j^{-k} A[j, i]. \tag{3.1}$$

If partition elements are equally sized then B_1 is left stochastic, i.e., $\sum_i B_1[i, j] = 1$ for each j. In general there exists a diagonal matrix D such that $D^{-1} B_1 D$ is left stochastic [13, §9.3].

For simplicity, in the following theorem we additionally suppose that f is topologically transitive. This means that there exists[4] a unique f-invariant probability measure which

[1] We consider only the transformations f which are orientation preserving. See Remark 3.7 concerning the general case $|f'| \geq \lambda$.

[2] It is also called the *incidence* matrix [13].

[3] We use throughout the convention $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$.

[4] The existence of these invariant measures is well known in this context and also follows from the results later in this section.

is absolutely continuous with respect to Lebesgue (denoted μ_{SRB}) and a unique measure of maximal entropy (also known as the Bowen-Margulis measure) (denoted μ_{BM}). We let h_{top} denote the topological entropy and $\mathcal{C}^\infty(\mathcal{I})$ denote the set of functions on I which are $\mathcal{C}^\infty$ when restricted to each I_j. We can now state a result concerning Ruelle-Pollicott resonances.

Theorem 3.1. *There exists a set of complex numbers $\Xi_1 = \{\xi_1, \xi_2, \cdots\}$ and, for each $\xi_i \in \Xi_1$, an associated integer[5] m_i such that, for any $\phi, \varphi \in \mathcal{C}^\infty(\mathcal{I})$ and $\epsilon > 0$ there is an asymptotic expansion*

$$\int_I \phi \cdot \varphi \circ f^n \, d\mu_{\text{SRB}} = \sum_{\xi_i \in \Xi_1 : |\xi_i| \geq \epsilon} \sum_{k=0}^{m_i - 1} \xi_i^n n^k C_{i,k}(\phi, \varphi) + o(\epsilon^n)$$

where $C_{i,k}(\phi, \varphi)$ are finite rank and non-zero bilinear functions of ϕ, φ.
The set Ξ_1 is equal (as a subset of $\mathbb{C}$) to[6] $\bigcup_{l=1}^\infty \sigma(B_l)$ and the equality holds also for the total multiplicity of each eigenvalue.[7]
Similarly there exists a set of complex numbers $\Xi_0 = \{\xi_1, \xi_2, \cdots\}$ and for each $\xi_i \in \Xi_0$ an associated integer m_i such that, for any $\phi, \varphi \in \mathcal{C}^\infty(\mathcal{I})$ and $\epsilon > 0$ there is an asymptotic expansion

$$\int_I \phi \cdot \varphi \circ f^n \, d\mu_{\text{BM}} = e^{-nh_{top}} \sum_{\xi_i \in \Xi_0 : |\xi_i| \geq \epsilon} \sum_{k=0}^{m_i - 1} \xi_i^n n^k C'_{i,k}(\phi, \varphi) + o(\epsilon^n)$$

where $C'_{i,k}(\phi, \varphi)$ are finite rank and non-zero bilinear functions of ϕ, φ. The set Ξ_0 is equal (as a subset of $\mathbb{C}$) to $\bigcup_{l=0}^\infty \sigma(B_l)$ and the equality holds also for the total multiplicity of each eigenvalue.

The proof of the above Theorem is included towards the end of the section and follows from a significantly stronger result (Theorem 3.4), described in terms of transfer operators, that needs some further preliminaries to be properly stated.

Remark 3.2. The assumption of topological transitivity means that B_1 is irreducible. Since also $D^{-1} B_1 D$ is left stochastic for some diagonal matrix D it follows that 1 is the leading eigenvalue of B_1 and this eigenvalue has multiplicity 1. Moreover $C_{1,0}(\phi, \varphi) = \int \phi \, d\mu_{\text{SRB}} \int \varphi \, d\mu_{\text{SRB}}$.

Remark 3.3. In the case where f has the form $x \mapsto \kappa x \mod 1$ for some $\kappa \in \{2, 3, \ldots\}$ we could consider f as a smooth map of the circle. In this case, restricting our attention to observables which are smooth on the circle, the set of Ruelle-Pollicott resonances would reduce[8] to $\{0\}$. However, studying the same systems for observables that are smooth on the interval, we see a much more interesting spectrum, see Remark 3.15.

Observe that for any $r \geq 0$, the Sobolev space $W^{r,1}(I)$ is the set of all $h \in L^1(I)$ such that h and all of its weak derivatives up to the r'th belong to $L^1(I)$. Consider, for any $r \geq 0$, the space $W^{r,1}(\mathcal{I})$ which is the set of all $h \in L^1(I)$ such that, for each i, the restriction of h to I_i is in $W^{r,1}(I_i)$. For convenience we write h' and $h^{(l)}$ to mean the weak derivative and l weak derivative respectively of h restricted to $\mathcal{I}$. For each $r \in \mathbb{N}_0$ the space $W^{r,1}(\mathcal{I})$ is a Banach space equipped with the norm

$$\|h\|_{r,1} = \sum_{l=0}^{r} \int_{\mathcal{I}} |h^{(l)}(x)| dx.$$

[5]The integer m_i is the Jordan block dimension. A given ξ_i might be repeated in Ξ_1 according to the geometric multiplicity.

[6]The spectrum of a matrix is denoted by σ.

[7]More can be said about the multiplicities and Jordan blocks, see Theorem 3.4 and Remark 3.14.

[8]This can be seen by considering the action of the dynamics on Fourier series.

In the following, to simply notation, we will write W_r for $W^{r,1}(\mathcal{I})$ and we will write $\|\cdot\|_r$ for $\|\cdot\|_{r,1}$. Observe that W_0 coincides with $L^1(I)$.

Since, by assumption, $f|_{I_j}$ is invertible on its range, let us call g_j its inverse ($g_j := f|_{I_j}{}^{-1}$). The domain of g_j is the interval $f(I_j)$ which might not be equal to the unit interval. If $f(I_j) = (0,1)$ for all j then f is said to be a *full branch* map. We can now define our main objects of investigation: the transfer operators. For all $k \in \mathbb{N}_0$, $h \in L^1(I)$ and $x \in I_i$ we define[9]

$$\mathcal{L}_k h(x) := \sum_{y \in f^{-1}(x)} \frac{h(y)}{[f'(y)]^k} = \sum_j B_k[i,j]\ h \circ g_j(x).$$

Since f preserves the Markov partition, composition with an affine transformation preserves Sobolev space and the sum consists of a finite number of terms it follows that these operators are well defined as operators $\mathcal{L}_k : W_r \to W_r$. Similarly they are well defined, by this same formula, on $\mathcal{C}^r(\mathcal{I})$.

Observe that $\mathcal{L}_1$ coincides with the usual transfer operator: the dual of the Koopman operator.

We define $\mathfrak{P}_r(\mathcal{I})$ to be the set of polynomial functions[10] of degree r on each interval I_j. Since f is piecewise affine, the space $\mathfrak{P}_r(\mathcal{I})$ is invariant under $\mathcal{L}_k$ for each $r, k \in \mathbb{N}_0$. Thus, it is natural to consider the finite rank operator $\mathcal{L}_k|_{\mathfrak{P}_r(\mathcal{I})}$.

Theorem 3.4. *Let $k \in \mathbb{N}_0$, $r \in \mathbb{N}$. There exists a projector $\Pi_{k,r} : W_r \to \mathfrak{P}_r(\mathcal{I}) \subset W_r$ such that the spectral radius of $\mathcal{L}_k(\mathbb{1} - \Pi_{k,r})$ on W_r is not greater than $\lambda^{-(k+r-1)}$. Moreover*

$$\sigma\big(\mathcal{L}_k|_{\mathfrak{P}_r(\mathcal{I})}\big) = \bigcup_{l=0}^{r} \sigma(B_{k+l})$$

and the multiplicity of each eigenvalue $\xi \in \sigma\big(\mathcal{L}_k|_{\mathfrak{P}_r(\mathcal{I})}\big)$ is equal to the sum of the multiplicities of ξ as eigenvalues of B_{k+l}, $l \in \{0, \dots, r\}$.

The remainder of this section is devoted to the proofs of the two above theorems.

Remark 3.5. For each $r \in \mathbb{N}$ this result tells nothing about the spectrum of $\mathcal{L}_k$ within the disk $\{|z| \le \lambda^{-(k+r-1)}\}$. Indeed we know that this disk contains the essential spectrum (see [26]).

The next equality is our key observation. Albeit very simple, the rest of the paper relies on it and variants thereof.

Lemma 3.6. *For all $k, r \in \mathbb{N}_0$, $h \in W_r$ and $l \in \{0, \dots, r\}$,*

$$(\mathcal{L}_k h)^{(l)} = \mathcal{L}_{k+l} h^{(l)}.$$

Proof. Fix $k, r \in \mathbb{N}_0$. The claimed equality holds trivially in the case $l = 0$. Observe that, by chain rule, for all $x \in \mathcal{I}_i$, $h \in \mathcal{C}^\infty(\mathcal{I})$,

$$(\mathcal{L}_k h)'(x) = \sum_j \lambda_j\ B_k[i,j] h' \circ g_j(x) = \sum_j B_{k+1}[i,j] h' \circ g_j(x) = \mathcal{L}_{k+1} h'(x).$$

If we assume that, for some $l \ge 0$, the claimed equality holds, i.e., for all $h \in \mathcal{C}^\infty(\mathcal{I})$,

$$(\mathcal{L}_k h)^{(l)}(x) = \mathcal{L}_{k+l} h^{(l)}(x),$$

[9] The second sum here is understood in the sense that, when the summands are defined on a subset of the full integral, they are extended to the full interval by taking the value zero where not defined.

[10] Studying the action on polynomials was also used for the vertical direction in the pseudo Anosov case [36].

then, using the previous observation,

$$(\mathcal{L}_k h)^{(l+1)}(x) = (\mathcal{L}_{k+l} h^{(l)})'(x) = (\mathcal{L}_{k+l+1} h^{(l+1)})(x).$$

The equality for all l follows by induction. Using the density of $\mathcal{C}^\infty(\mathcal{I})$ in W_r we obtain the result for $h \in W_r$. □

Remark 3.7. In general we could allow the λ_j to be positive or negative. If $\mathcal{L}_1$ coincides with the operator associated to the SRB measure the derivative λ_j occurs with absolute value in the formula. However, as is clear from the proof of the above lemma, when the derivative occurs as a result of differentiating the sign of the derivative remains. This means that, if we are interested in μ_{BM} then we should consider $B_k[i,j] := \lambda_j^{-k} A[j,i]$ but, if we are interested in μ_{SRB}, then we should consider $B_k[i,j] := \lambda_j^{-(k-1)} \left|\lambda_j\right|^{-1} A[j,i]$.

To proceed we now prove a set of Lasota-Yorke inequalities for the operators $\mathcal{L}_k : W_r \to W_r$.

Let $\Gamma_0 := \|f'\|_{L^\infty}$ and, for all $k \in \mathbb{N}$, let $\Gamma_k := \lambda^{-(k-1)}$.

Lemma 3.8. *Let $k \in \mathbb{N}_0$, $r \in \mathbb{N}$. For all $h \in W_r$,*

$$\begin{aligned}
\|\mathcal{L}_k h\|_r &\leq \Gamma_k \|h\|_r \\
\|\mathcal{L}_k h\|_r &\leq \lambda^{-(k+r-1)} \|h\|_r + \Gamma_k \|h\|_{r-1}.
\end{aligned}$$

The first inequality also holds in the case $r = 0$.

Proof. We start by considering the case $k \in \mathbb{N}$. Let $h \in W_r$. By definition of $\|\cdot\|_r$ and Lemma 3.6,

$$\begin{aligned}
\|\mathcal{L}_k h\|_r &= \sum_{l=0}^{r} \int_{\mathcal{I}} |(\mathcal{L}_k h)^{(l)}(x)| dx = \sum_{l=0}^{r} \int_{\mathcal{I}} |\mathcal{L}_{k+l} h^{(l)}(x)| dx \\
&\leq \sum_{l=0}^{r} \lambda^{-(k+l-1)} \int_{\mathcal{I}} \mathcal{L}_1 |h^{(l)}(x)| dx.
\end{aligned}$$

Since, by the obvious change of variables, $\int_{\mathcal{I}} \mathcal{L}_1 |h^{(l)}(x)| dx = \int_{\mathcal{I}} |h^{(l)}(x)| dx$ the above implies that, for all $r \in \mathbb{N}_0$,

$$\|\mathcal{L}_k h\|_r \leq \sum_{l=0}^{r} \lambda^{-(k+l-1)} \int_{\mathcal{I}} |h^{(l)}(x)| dx. \tag{3.2}$$

That is, $\|\mathcal{L}_k h\|_r \leq \lambda^{-(k-1)} \|h\|_r$ as required to prove the first inequality. Moreover, when $r \geq 1$, the above (3.2) implies that (here we separate the term $l = r$ from the rest of the sum)

$$\begin{aligned}
\|\mathcal{L}_k h\|_r &\leq \lambda^{-(k+r-1)} \int_{\mathcal{I}} |h^{(r)}(x)| dx + \lambda^{-(k-1)} \sum_{l=0}^{r-1} \int_{\mathcal{I}} |h^{(l)}(x)| dx \\
&\leq \lambda^{-(k+r-1)} \|h\|_r + \lambda^{-(k-1)} \|h\|_{r-1}
\end{aligned}$$

as required by the second estimate.

To conclude we must consider the case $k = 0$. First observe that, for any $h \in \mathcal{C}^\infty(\mathcal{I})$,

$$\int_{\mathcal{I}} |\mathcal{L}_0 h|(x)\ dx = \int_{\mathcal{I}} |\mathcal{L}_1 f' h|(x) \leq \int_{\mathcal{I}} \left[\mathcal{L}_1 |f'|\, |h|\right](x) \leq \|f'\|_{L^\infty} \int_{\mathcal{I}} |h|(x)\ dx.$$

Similar to the proof in the case $k \in \mathbb{N}$, by definition of the norm and Lemma 3.6,

$$\|\mathcal{L}_0 h\|_r = \sum_{l=0}^{r} \int_{\mathcal{I}} |(\mathcal{L}_0 h)^{(l)}(x)|dx = \sum_{l=0}^{r} \int_{\mathcal{I}} |\mathcal{L}_l h^{(l)}(x)|dx.$$

This means that, for all $r \in \mathbb{N}_0$ (recall that $\Gamma_0 = \|f'\|_{L^\infty}$),

$$\|\mathcal{L}_0 h\|_r \leq \sum_{l=0}^{r} \lambda^{-l} \int_{\mathcal{I}} |\mathcal{L}_0 h^{(l)}(x)|dx \leq \Gamma_0 \sum_{l=0}^{r} \int_{\mathcal{I}} |h^{(l)}(x)|dx$$

and so proves the first inequality. On the other hand, now assuming that $r \geq 1$,

$$\begin{aligned} \|\mathcal{L}_0 h\|_r &= \int_{\mathcal{I}} |\mathcal{L}_r h^{(r)}(x)|dx + \sum_{l=0}^{r-1} \int_{\mathcal{I}} |\mathcal{L}_l h^{(l)}(x)|dx \\ &\leq \lambda^{-(r-1)} \int_{\mathcal{I}} |\mathcal{L}_1 h^{(r)}(x)|dx + \sum_{l=0}^{r-1} \int_{\mathcal{I}} |\mathcal{L}_0 h^{(l)}(x)|dx. \end{aligned}$$

Consequently

$$\|\mathcal{L}_0 h\|_r \leq \lambda^{-(r-1)} \int_{\mathcal{I}} |h^{(r)}(x)|dx + \Gamma_0 \sum_{l=0}^{r-1} \int_{\mathcal{I}} |h^{(l)}(x)|dx.$$

Thus, $\|\mathcal{L}_0 h\|_r \leq \lambda^{-(r-1)} \|h\|_r + \Gamma_0 \|h\|_{r-1}$, as required. □

Lemma 3.9. *Let $k \in \mathbb{N}_0$, $r \in \mathbb{N}$. The operator $\mathcal{L}_k$ acting on W_r has spectral radius bounded by Γ_k and essential spectral radius bounded by $\lambda^{-(k+r-1)}$.*

Proof. The first inequality of Lemma 3.8 implies that the spectral radius is bounded by Γ_k. For all $r \in \mathbb{N}$, W_r is compactly embedded in W_{r-1}. This means the Lasota-Yorke inequalities of Lemma 3.8 imply, by Theorem 2.23, that the essential spectral radius of $\mathcal{L}_k$ is bounded by $\lambda^{-(k+r-1)}$. □

Remark 3.10. The above estimate of the spectral radius will often not be optimal but it suffices for our present purposes. Subsequently we will improve this estimate by proving a connection of the spectrum of the transfer operators with the spectrum of the matrices B_k.

For convenience we use the notation $\mathcal{D} : h \mapsto h'$. For any $k \in \mathbb{N}$, $\nu \in \mathbb{C}$, let $E_k(\nu)$ denote the generalised eigenspace for $\mathcal{L}_k$ associated to the eigenvalue ν. I.e., $E_k(\nu)$ is the set of h such that $(\mathcal{L}_k - \nu)^m h = 0$ for some $m \in \mathbb{N}$. An immediate consequence of Lemma 3.6 is the following commutation relation: For any $l, k, m \in \mathbb{N}_0$, $\nu \in \mathbb{C}$, $h \in W_r$

$$\mathcal{D}^l \circ (\mathcal{L}_k - \nu)^m h = (\mathcal{L}_{k+l} - \nu)^m \circ \mathcal{D}^l h.$$

This in turn means that

$$\mathcal{D}^l E_k(\nu) \subset E_{k+l}(\nu). \tag{3.3}$$

Proof of the first statement of Theorem 3.4. Let $k \in \mathbb{N}_0$, $r \in \mathbb{N}$. According to Lemma 3.9 the essential spectral radius of $\mathcal{L}_k : W_r \to W_r$ is not greater that $\lambda^{-(k+r-1)}$. Fix some arbitrarily small $\epsilon > 0$ and define $\mathcal{H}_{k,r} := \{\nu \in \sigma_{W_r}(\mathcal{L}_k), |\nu| > \lambda^{-(k+r-1)} + \epsilon\}$.[11] For each $\nu \in \mathcal{H}_{k,r}$ let P_ν denote the associated spectral projector and hence let $\Pi_{k,r} := \sum_{\nu \in \mathcal{H}_{k,r}} P_\nu$. Consequently $\mathcal{L}_k - \mathcal{L}_k \circ \Pi_{k,r} : W_r \to W_r$ has spectral radius not greater than $\lambda^{-(k+r-1)} + \epsilon$. For any $l \in \mathbb{N}$

[11] We denote by $\sigma_B(\mathcal{L})$ the spectrum of an operator $\mathcal{L}$ acting on a Banach space B.

Lemma 3.9 gives an upper bound of $\lambda^{-(l-1)}$ for the spectral radius of $\mathcal{L}_l : W_1 \to W_1$ and so $E_l(\nu) = \{0\}$ whenever $|\nu| > \lambda^{-(l-1)}$. As observed above (3.3), differentiating r times takes the generalised eigenspace $E_k(\nu)$ to the generalised eigenspace $E_{k+r}(\nu)$ of the operator $\mathcal{L}_{k+r}$. However $E_{k+r}(\nu) = \{0\}$ since $|\nu| > \lambda^{-(k+r-1)}$. This means that $E_k(\nu) \subset \mathfrak{P}_r(\mathcal{I})$ whenever $\nu \in \mathcal{H}_{k,r}$ and so we have shown that the image of $\Pi_{k,r}$ is contained in $\cup_{\nu\in\mathcal{H}_{k,r}} E_k(\nu) \subset \mathfrak{P}_r(\mathcal{I})$. The claim follows by the arbitrariness of ϵ. □

We can identify $\mathbb{R}^N$ with $\mathfrak{P}_0(\mathcal{I})$, the set of functions that are constant on each partition element, in the sense that we associate the function[12] $\sum_i a_i \mathbf{1}_{I_i} \in \mathfrak{P}_0(\mathcal{I})$ to each $a = (a_i) \in \mathbb{R}^N$.

Let $r \in \mathbb{N}$. The space $(\mathbb{R}^N)^{(r+1)}$ is identified with $\mathfrak{P}_r(\mathcal{I})$ as follows. We use the notation $(a^0, a^1, \dots, a^r) \in (\mathbb{R}^N)^{(r+1)}$ where $a^j = (a^j_1, a^j_2, \dots, a^j_N)$ for each j. Let[13] $\mathcal{J} : \mathbb{R}^{N(r+1)} \to \mathfrak{P}_r(\mathcal{I})$,

$$\mathcal{J}(a^0, \dots, a^r) : x \mapsto \sum_{l=0}^{r} x^l \sum_{j=1}^{N} a^l_j \mathbf{1}_{I_j}(x).$$

Observe that $\mathcal{J} : \mathbb{R}^{N(r+1)} \to \mathfrak{P}_r(\mathcal{I})$ is onto and invertible.

For any $k \in \mathbb{N}_0$, $r \in \mathbb{N}$ we define the $N(r+1) \times N(r+1)$ matrix

$$\mathcal{T}_{k,r} := \mathcal{J}^{-1} \circ \mathcal{L}_k|_{\mathfrak{P}_r(\mathcal{I})} \circ \mathcal{J}.$$

In order to understand the spectrum of $\mathcal{L}_k|_{\mathfrak{P}_r(\mathcal{I})}$ it suffices to study the spectrum of the matrix $\mathcal{T}_{k,r}$.

Remark 3.11. It is not required for the present work but the inverse has the form $\mathcal{J}^{-1} : h \mapsto (a^0, a^1, \dots, a^r)$ where $a^r = \frac{1}{r!} h^{(r)}$ and, iteratively for $l \in \{r-1, r-2, \dots, 0\}$,

$$a^l = \frac{1}{l!}\Big(h - \sum_{j=l+1}^{r} \mathbf{x}^j a^j\Big)^{(l)}.$$

This is well defined since the terms $h^{(r)}$ and[14] $h - \sum_{j=l+1}^{r} \mathbf{x}^j a^j$ are guaranteed to be piecewise constant by construction. We prove that the above formula coincides with $\mathcal{J}^{-1}$ as follows. Let $(a^0, a^1, \dots, a^r) \in \mathbb{R}^{N(r+1)}$ and consider $(b^0, b^1, \dots, b^r) = \mathcal{J}^{-1} \circ \mathcal{J}(a^0, a^1, \dots, a^r)$. I.e.,

$$(b^0, b^1, \dots, b^r) = \mathcal{J}^{-1}\Big(\sum_{l=0}^{r} \mathbf{x}^l a^l\Big).$$

Since $\left(\sum_{l=0}^{r} \mathbf{x}^l a^l\right)^{(r)} = r! a^r$ we see that $b^r = a^r$ as required. Now suppose that we have already shown that $b^j = a^j$ for all $j \in \{l+1, \dots, r\}$. This means that

$$b^l = \frac{1}{l!}\pi\Big(\mathcal{J}(a^0, a^1, \dots, a^r) - \sum_{j=l+1}^{r} \mathbf{x}^l a^l\Big)^{(l)} = \frac{1}{l!}\pi\Big(\sum_{j=0}^{l} \mathbf{x}^j a^j\Big)^{(l)} = a^l.$$

Lemma 3.12. *Let $k \in \mathbb{N}_0$, $r \in \mathbb{N}$. Then $\mathcal{T}_{k,r}$ has lower block triangular form*

$$\mathcal{T}_{k,r} = \begin{pmatrix} B_k & & & 0 \\ F_{1,0} & B_{k+1} & & \\ \vdots & \vdots & \ddots & \\ F_{r,0} & F_{r,1} & \dots & B_{k+r} \end{pmatrix},$$

[12] The symbol $\mathbf{1}_A$ denotes the characteristic function of the set A.

[13] Abusing notation we will often write the same symbol for $a \in \mathbb{R}^N$ and the corresponding $a \in \mathfrak{P}_0(\mathcal{I})$ with the interpretation given by context.

[14] We use the notation $\mathbf{x}^n$ to denote the function $x \mapsto x^n$.

where the matrices on the diagonal are the ones previously introduced in (3.1).

Before proving the above lemma it is convenient to introduce some further notation. For each j let $q_j := f(p_j^+)$ (i.e., $\lim_{\epsilon\to 0} f(p_j+\epsilon)$). Observe that, for all $x \in I_j$, $f(x) = \lambda_j(x-p_j)+q_j$. Consequently, for all $x \in f(I_j)$,

$$g_j(x) = (x-q_j)\lambda_j^{-1} + p_j. \tag{3.4}$$

Proof of Lemma 3.12. Fix $k \in \mathbb{N}_0$, $r \in \mathbb{N}$. For any $l \in \{0,\dots,r\}$ we consider $(a^0,a^1,\dots,a^r) \in \mathbb{R}^{N(r+1)}$ and suppose that $a^j = 0$ for all $j \neq l$. This means that $\mathcal{J}(a^0,a^1,\dots,a^r) = \mathbf{x}^l a$. We wish to compute $\mathcal{J}^{-1}\circ\mathcal{L}_k\circ\mathcal{J}(a^0,a^1,\dots,a^r) = \mathcal{J}^{-1}\circ\mathcal{L}_k(\mathbf{x}^l a)$. Using formula (3.4) for the inverse

$$\begin{aligned}\mathcal{L}_k(\mathbf{x}^l a)(x) &= \sum_{i,j} \mathbf{1}_{I_i}(x) B_k[i,j] a_j \left(x\lambda_j^{-1} - q_j\lambda_j^{-1} + p_j\right)^l \\ &= \sum_{i,j} \mathbf{1}_{I_i}(x) B_{l+k}[i,j] a_j x^l + \rho(x),\end{aligned} \tag{3.5}$$

where $\rho \in \mathfrak{P}_{l-1}(\mathcal{I})$. I.e., $\mathcal{L}_k(\mathbf{x}^l a^l) = \mathbf{x}^l B_{k+l} a^l + \rho$, where $\rho \in \mathfrak{P}_{l-1}(\mathcal{I})$. This proves that $\mathcal{T}_{k,r}$ has lower diagonal block form and that the diagonal elements of the block matrix are the B_{k+l}. ☐

Remark 3.13. The exact form of the matrices $F_{i,j}$ which appear below the diagonal in Lemma 3.12 are superfluous to our present argument and we won't identify them further. If they were required they can be obtained from the above details (3.5) and Remark 3.11.

Proof of Theorem 3.4. The first statement of the theorem was proven above, it remains to prove the second statement. Recall the lower triangular block form of $\mathcal{T}_{k,r}$ as shown in Lemma 3.12. We can assume without loss of generality that each B_{k+l} is in lower triangular form. If a matrix is in triangular form then the values on the diagonal are the eigenvalues repeated according to multiplicity. That each B_{k+l} is in triangular form means that the $N(r+1)\times N(r+1)$ matrix $\mathcal{T}_{k,r}$ is in triangular form. Moreover the diagonal is the union of the diagonals of the B_{k+l}. This implies the claimed correspondence of the eigenvalues of $\mathcal{T}_{k,r}$ and the union of the set of eigenvalues of the $\{B_{k+l}\}_{l=0}^r$, including correspondence in multiplicity. ☐

Remark 3.14. The lower triangular block form shown in Lemma 3.12 and the argument of the above proof further implies that, if some B_{k+l} has a Jordan block of dimension $m \in \mathbb{N}$, then $\mathcal{T}_{k,r}$ has a corresponding Jordan block of dimension m or greater. On the other hand $\mathcal{T}_{k,r}$ has the possibility to have a Jordan block of greater dimension if a given eigenvalue appears in more than one of the B_{k+l}.

Proof of Theorem 3.1. Fix $k \in \mathbb{N}_0$. For each $r \in \mathbb{N}$ consider the finite set of eigenvalues

$$\{\xi_j\}_{j=0}^{K_r} = \sigma_{W_r}(\mathcal{L}_k) \setminus \{|z| \le \lambda^{-(k+r-1)}\}$$

described by Theorem 3.4. We define as usual the corresponding eigen projectors $\{\Pi_j : W_r \to W_r\}_{j=0}^{K_r}$ and eigen nilpotents $\{Q_j : W_r \to W_r\}_{j=0}^{K_r}$ which satisfy the commutation relations: $\Pi_j\Pi_k = \delta_{j,k}$, $\Pi_j Q_k = Q_k\Pi_j = \delta_{j,k}Q_k$. Let $S_r := \mathbb{1} - (\Pi_1+\Pi_2+\cdots+\Pi_{K_r})$ and observe that $\mathcal{L}_k S_r$ has spectral radius not greater than $\lambda^{-(k+r-1)}$. This means that the operator $\mathcal{L}_k : W_r \to W_r$ satisfies the decomposition

$$\mathcal{L}_k = \sum_{j=1}^{K_r} (\xi_j\Pi_j + Q_j) + \mathcal{L}_k S_r. \tag{3.6}$$

Further observe that each operator we define remains defined by the same formula on W_r for any r sufficiently large.

Now let us recall the connection between the transfer operators and invariant measures (see [61] or [6] for full details). For each $k \in \{0,1\}$ there exists $h_k \in W_r$ (*the invariant density*), a probability measure ν_k (*the conformal measure*), $\gamma_k > 0$ (*the spectral radius*) and a probability measure μ_k defined as $\mu_k(\varphi) := \nu_k(h_k\varphi)$ (*the invariant measure*). Moreover $\nu_k(\mathcal{L}_k\varphi) = \gamma_k\nu_k(\varphi)$ and $\mu_k(\varphi \circ f) = \mu_k(\varphi)$.

In our present setting μ_0 is the measure of maximal entropy μ_{BM} and μ_1 is the SRB measure μ_{SRB}. Furthermore $\ln \gamma_0$ is equal to the topological entropy, $\gamma_1 = 1$ and ν_1 coincides with Lebesgue measure.

Continuing for $k \in \{0,1\}$ we observe that

$$\begin{aligned}\int_I \phi \cdot \varphi \circ f^n \; d\mu_k &= \int_I (\phi \cdot h_k)(x) \cdot \varphi \circ f^n(x) \; d\nu_k(x) \\ &= \gamma_k^{-n} \int_I \mathcal{L}_k^n(\phi \cdot h_k)(x) \cdot \varphi(x) \; d\nu_k(x)\end{aligned}$$

We then combine this formula with the spectral decomposition above (3.6) to produce the asymptotic expansion required. □

Remark 3.15. If f is full branch, the matrices B_{k+l} are such that all the entries in any column j is equal to $\lambda_j^{-(k+l)}$. The spectrum of this type of matrix is the union of zero and the sum of entries on different columns. Consequently Theorem 3.4 implies that, outside of the disk $\{|\nu| \le \lambda^{-(k+r-1)}\}$, the spectrum of $\mathcal{L}_k : W_r \to W_r$ is equal to $\{\xi_0, \ldots, \xi_{r-1}\}$ where $\xi_l := \sum_{j=1}^N \lambda_j^{-(k+l)}$.

- In the case $k = 0$ we obtain $\xi_0 = \sum_{j=1}^N \lambda_j^0 = N$;
- In the case $k = 1$ we see that $\xi_0 = \sum_{j=1}^N \lambda_j^{-1} = \sum_{j=1}^N |I_j| = 1$.

Remark 3.16. Observe that B_0 is the transpose of A and that, for any Markov map as considered in the present section, the logarithm of the spectral radius of B_0 is equal [21, §2.1] to the topological entropy.

Remark 3.17. Consistent with previous notation, $\mathcal{C}^r(\mathcal{I})$ denotes the functions which are $\mathcal{C}^r$ smooth on each partition element I_j. In this section we used Sobolev spaces W_r but, with a slightly more complex argument, we could equally well have worked with $\mathcal{C}^r(\mathcal{I})$.

3.0.1 A Jordan block example

In the following we construct an example of a Markov expanding map such that B_1 has a Jordan block of dimension two (strangely enough we are not aware of a published example that shows the existence of Jordan blocks in the spectrum of a transfer operator).

Let $I_1 = (0, \frac{1}{4})$, $I_2 = (\frac{1}{4}, \frac{1}{2})$, $I_3 = (\frac{1}{2}, \frac{3}{4})$, $I_4 = (\frac{3}{4}, 1)$ and let $f : I \to I$ be as shown in Figure 3.1, defined by

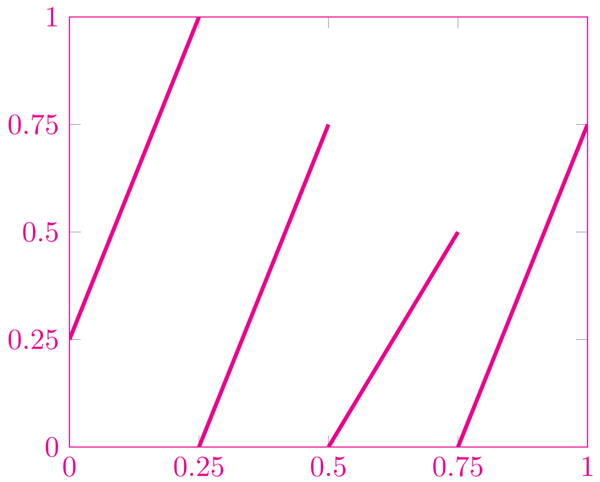
$$f(x) := \begin{cases} 3x + \frac{1}{4} & \text{if } x \in I_1 \\ 3(x - \frac{1}{4}) & \text{if } x \in I_2 \\ 2(x - \frac{1}{2}) & \text{if } x \in I_3 \\ 3(x - \frac{3}{4}) & \text{if } x \in I_4. \end{cases}$$

This means that

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & 0 \end{pmatrix}.$$

The matrix B_1 has the eigenvalues $\{-\frac{1}{3}, 0, 1\}$ and the eigenvalue $-\frac{1}{3}$ has a Jordan block of

Figure 3.1: An expanding Markov map with a non-trivial Jordan block



dimension two. If we let

$$a_1 := \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad a_2 := \begin{pmatrix} 3 \\ 3 \\ -6 \\ 0 \end{pmatrix}, \quad a_3 := \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad a_4 := \begin{pmatrix} 9 \\ 12 \\ 8 \\ 3 \end{pmatrix},$$

then $B_1 a_1 = -\frac{1}{3} a_1$, $(B_1 + \frac{1}{3}\mathbb{1}) a_2 = a_1$, $B_1 a_3 = 0$ and $B_1 a_4 = a_4$. In particular $\{a_1, a_2\}$ span the generalised eigenspace associated to the eigenvalue $-\frac{1}{3}$. Since the essential spectral radius of $\mathcal{L}_1$, when acting on W_r, $r \geq 2$, is smaller than $1/4$, then $\mathcal{L}_1$, on such spaces, has a Jordan block in the point spectrum.

Remark 3.18. Another example of an affine expanding Markov map is the Baladi map studied in [25]. This is a system which exhibits non-trivial complex resonances. In the reference the connection between resonances and decay of correlation was considered and the outer set of resonances identified. Our results give a description of the full set of resonances for this system and the connection to the decay of correlations.

Remark 3.19. The simplest example that fits the framework of this chapter is the doubling map, $x \mapsto 2x \mod 1$ (a comprehensive investigation of the resonances can be found in [31, §3]). The eigenfunctions for the doubling map are the Bernoulli polynomials.

Chapter 4

Piecewise smooth full branch expanding maps

In this section we discuss the simplest non-linear case: full branch maps. For such maps there exists already some general quantitative results on the spectral gap, e.g., [60, Section 2], however they are not optimal, we will comment about the comparison case by case.

Let $f \in \mathcal{C}^r([0,1],[0,1])$, $r \geq 2$, be a full branched piecewise expanding map, $f' > \lambda > 1$. For $k \in \mathbb{N}_0$ let us consider the transfer operator

$$\mathcal{L}_k h(x) = \sum_{y \in f^{-1}(x)} \frac{h(y)}{[f'(y)]^k}.$$

Observe that $\mathcal{L}_0$ is the operator associated to the measure of maximal entropy while $\mathcal{L}_1$ is the operator associated to the SRB measure [6].[1]

The key fact we wish to leverage on, in analogy with Lemma 3.6, is the following formula

$$(\mathcal{L}_k h)' = \mathcal{L}_{k+1} h' + k\mathcal{L}_k(hD) \tag{4.1}$$

where

$$D = (\frac{1}{f'})'.$$

[2] Amazingly, the above formula yields several non trivial facts. To illustrate its power we start discussing the operator $\mathcal{L}_0$.

4.0.1 Measure of maximal entropy

Lemma 4.1. *If f is a N covering, then $\sigma_{\mathcal{C}^1}(\mathcal{L}_0) \subset \{N\} \cup \{z \in \mathbb{C} \ : \ |z| \leq 1\}$. Moreover, $\sigma_{\mathcal{C}^2}(\mathcal{L}_0) \cap \{z \in \mathcal{C} \ : \ |z| \geq \lambda^{-1}\} = \{N\} \cup (\sigma_{\mathcal{C}^1}(\mathcal{L}_1) \cap \{z \in \mathbb{C} \ : \ |z| \geq \lambda^{-1}\})$*

[1] For the measure of maximal entropy see also the beginning of section 5 where it is explained in a more general setting.

[2] Here, as in the previous section, the derivative is taken only at the smoothness points of f.

Proof. Note that $\mathcal{L}_0 1 = N$, so $N \in \sigma(\mathcal{L}_0)$. By (4.1) and a direct computation[3]

$$\|\mathcal{L}_0 h\|_\infty \le N\|h\|_\infty,$$
$$\|(\mathcal{L}_0^n h)'\|_\infty \le \|\mathcal{L}_1^n h'\|_\infty \le C\|h'\|_\infty.$$

By usual arguments this implies that the essential spectral radius of $\mathcal{L}_0$, acting on $\mathcal{C}^1$, is one. On the other hand if $\mathcal{L}_0 h = \nu h$, with $|\nu| > 1$ we have, for all $n \in \mathbb{N}$,

$$\nu^n h' = (\mathcal{L}_0^n h)' = \mathcal{L}_1^n h'$$

which, since $\|\mathcal{L}_1^n\|_{L^1} = 1$, implies $|h'| = 0$, that is, h must be constant on the monotonicity intervals of f. But since $h(x) = \nu^{-1} \sum_{y \in F^{-1}(x)} h(y)$, it follows that h is constant, thus $\nu = N$.

To conclude observe that on the one hand, if $\mathcal{L}_0 H = \nu H$, $H \in \mathcal{C}^2$, then $\mathcal{L}_1 H' = \nu H'$. On the other hand, if $\mathcal{L}_1 h = \nu h$, $\nu \neq N$,$h \in \mathcal{C}^1$, set $H_c(x) = \int_0^x h(y)dy + c$. We have $H_c \in \mathcal{C}^2$ and

$$((\nu - \mathcal{L}_0)H_c)' = (\nu - \mathcal{L}_1)h = 0$$

Thus there exists constants α such that $\alpha = (\nu - \mathcal{L}_0)H_0$, hence

$$(\nu - \mathcal{L}_0)H_c = (\nu - \mathcal{L}_0)H_0 + (\nu - \mathcal{L}_0)c = (\nu - \mathcal{L}_0)H_0 + (\nu - N)c = \alpha + (\nu - N)c.$$

Thus, choosing $c = -(\nu - N)^{-1}\alpha$, we have $\mathcal{L}_0 H_c = \nu H_c$. The result follows since the essential spectrum of $\mathcal{L}_1$, when acting on $\mathcal{C}^1$, is bounded by λ^{-1}.

□

Remark 4.2. Note that the proof of Lemma 4.1 implies that $\mathcal{L}_0 h = \int h \, d\mu_{\text{BM}} + Qh$, where μ_{BM} is the measure of maximal entropy, and $\|Q^n\|_{\mathcal{C}^1} \le C$. That is, $\mathcal{L}_0$ has a spectral gap $N - 1$ while the Hilbert metric technique can yield, at the very best, a spectral gap $N - \lambda$, see [60].

The above shows that the spectrum of $\mathcal{L}_0$ is largely determined by the spectrum of $\mathcal{L}_1$. Hence, before continuing our investigation of the spectrum of $\mathcal{L}_0$, it is necessary to undertake an investigation of the spectrum of $\mathcal{L}_1$.

4.0.2 The SRB measure

Note that the vector space $\mathbb{V} = \{h \in \mathcal{C}^1 \; : \; \int_0^1 h = 0\}$ is invariant under $\mathcal{L}_1$, we can thus restrict $\mathcal{L}_1$ to $\mathbb{V}$. If we define

$$\phi(g)(x) = \int_0^x g(y)dy - \int_0^1 (1-y)g(y)dy = \int_0^x yg(y)dy + \int_x^1 (y-1)g(y)dy, \tag{4.2}$$

then $\phi : \mathcal{C}^0 \to \mathbb{V}$ and $\phi(h') = h$ for all $h \in \mathbb{V}$. Thus, for each $h \in \mathbb{V}$,

$$(\mathcal{L}_1 h)' = \mathcal{L}_2 h' + \mathcal{L}_1(\phi(h')D) =: \mathcal{L}_\star(h'). \tag{4.3}$$

The relevance of the operator $\mathcal{L}_\star$ rests in the next Lemma.

[3]Note that

$$\left(\frac{1}{(f^n)'}\right)' = \sum_{k=0}^{n-1} \frac{D \circ f^k}{(f^{n-k-1})' \circ f^{k+1}}$$

thus $\left|\left(\frac{1}{(f^n)'}\right)'\right| \le \|D\|_\infty (1 - \lambda^{-1})^{-1}$. We can thus use formula (4.1) for f^n, rather than for f.

Lemma 4.3. *If $f \in \mathcal{C}^2([0,1],[0,1])$, then the spectrum of $\mathcal{L}_1 : \mathcal{C}^1 \to \mathcal{C}^1$ satisfies*

$$\sigma_{\mathcal{C}^1}(\mathcal{L}_1) \cap \{z \in \mathbb{C} \,:\, |z| > \lambda^{-1}\} = \{1\} \cup \sigma_{\mathcal{C}^0}(\mathcal{L}_\star) \cap \{z \in \mathbb{C} \,:\, |z| > \lambda^{-1}\}.$$

Proof. It is well known that the essential spectral radius of $\mathcal{L}_1$, when acting on $\mathcal{C}^1$ is bounded by λ^{-1}, hence we can restrict ourselves to the point spectrum.

Since $\int_0^1 \varphi \mathcal{L}_1 h = \int_0^1 (\varphi \circ f) \cdot h$, it follows that the Lebesgue measure is an eigenvector, with eigenvalue one, of the dual operator, and hence $1 \in \sigma_{\mathcal{C}^1}(\mathcal{L}_1)$. In addition, $\mathbb{V} = \{h \in \mathcal{C}^1 \,:\, \int_0^1 h = 0\}$ is invariant under $\mathcal{L}_1$. It follows that if $\mathcal{L}_1 h = \nu h$, $|\nu| > \lambda^{-1}$ and $h \in \mathcal{C}^1$, then $h' \in \mathcal{C}^0$ and (4.3) implies $\mathcal{L}_\star h' = \nu h'$. On the other hand if $g \in \mathcal{C}^0$ and $\mathcal{L}_\star g = \nu g$, $|\nu| > \lambda^{-1}$, then $h = \phi(g) \in \mathbb{V}$ and

$$(\mathcal{L}_1 h - \nu h)' = \mathcal{L}_\star g - \nu g = 0.$$

Hence, there exists a constant C such that $\mathcal{L}_1 h - \nu h = C$, but integrating we have $C = 0$, thus h is an eigenvector of $\mathcal{L}_1$. □

Remark 4.4. Note that the above Lemma holds verbatim with $W^{1,1}$ substituted to $\mathcal{C}^1$. In the following, we find more convenient to consider the spectrum of $\mathcal{L}_1$ when acting on $W^{1,1}$.

Lemma 4.5. *The norm of $\mathcal{L}_2$ on L^1 is bounded by λ^{-1}. The operator $\mathcal{L}_c(g) = \mathcal{L}_1(\phi(g)D)$, acting on L^1, is a compact operator. In addition, for all $g \in L^1$*

$$\begin{aligned}
\|\phi(g)\|_{L^1} &\le \frac{1}{2}\|g\|_{L^1} \\
\|\phi(g)\|_{L^\infty} &\le \|g\|_{L^1} \\
\|\phi(g)\|_{W^{1,1}} &\le \frac{3}{2}\|g\|_{L^1}.
\end{aligned}$$

Proof. Since

$$\|\mathcal{L}_2 h\|_{L^1} \le \lambda^{-1}\|\mathcal{L}_1 h\|_{L^1} \le \lambda^{-1}\|h\|_{L^1},$$

the first statement follows. Moreover,

$$\begin{aligned}
\int_0^1 |\phi(g)(x)| dx &\le 2\int_0^1 |g(y)| y(1-y) dy \le \frac{1}{2}\|g\|_{L^1}, \\
|\phi(g)(x)| &\le \int_0^1 |g(y)| dy = \|g\|_{L^1}.
\end{aligned}$$

Finally, $\|\phi(g)'\|_{L^1} \le \|g\|_{L^1}$ implies the last inequality and also that ϕ is compact, the compactness of $\mathcal{L}_c$ follows. □

Theorem 4.6. *Let us consider $\mathcal{L}_1$ as an operator acting on $W^{1,1}$, then $\sigma_{ess}(\mathcal{L}_1) \subset \{z \in \mathbb{C} \,:\, |z| \le \lambda^{-1}\}$. Moreover $\sigma(\mathcal{L}_1) \setminus \{1\} \subset \{z \in \mathbb{C} \,:\, |z| \le \tau\}$, where*

$$\tau = \lambda^{-1} + \int_0^1 \left| \left(\frac{1}{f'(y)} \right)' \right| dy = \lambda^{-1} + \|D\|_{L^1}.$$

Proof. If $\nu \in \mathbb{C}$ is such that $|\nu| > \lambda^{-1}$ and $\mathcal{L}_1 h = \nu h$, for some $h \in W^{1,1}$ with $\int h = 0$, then $\mathcal{L}_\star g = \nu g$, for $g = h'$. Then, recalling Lemma 4.5 ,

$$\begin{aligned}
|\nu| \|g\|_{L^1} &\le \lambda^{-1}\|g\|_{L^1} + \int_0^1 \left| \left(\frac{1}{f'(y)} \right)' \right| dy \; \|\phi(g)\|_{L^\infty} \\
&\le \left[\lambda^{-1} + \|D\|_{L^1}\right] \|g\|_{L^1}.
\end{aligned}$$

This proves the theorem since $h' = 0$ implies $h = 0$. □

The above lemma provides an upper bound for the spectral gap, but it is very unsatisfactory. First, such a bound is of interest only if $\tau < 1$ (for example, in the counterexample of Keller, Rugh [55] $\tau > 1$). Second, even if $\tau < 1$, it is unclear if there exists other point spectrum outside $\{z \in \mathbb{C} \,:\, |z| \le \lambda^{-1}\}$.

Remark 4.7. For $\mathcal{L}_1$ the Hilbert metric approach yields a bound of the spectral gap given by a rather cumbersome formula. However, if one considers the limit of large λ and small D, then, using [60, Lemma 2.3], one can check that the bound of the spectral gap cannot be better than $\lambda^{-1}(1 + 2\|D\|_\infty) + \|D\|_\infty$, which is worse than the one provided, in the same limit, by Theorem 4.6. However, for large D the bound of Theorem 4.6 is useless while [60, Lemma 2.3] provides an explicit, although rather poor, bound.

Very few results are known on the existence of point spectrum with the notable exception of cases when the map has been explicitly constructed to exhibit point spectrum [55] or when one restricts the map to the class of holomorphic maps, often of a special nature, as in [9, 77, 78]. No analytical technique is available to treat $\mathcal{C}^2$ open classes of maps. On the contrary a lot of work exists on the side of numerical computation, mainly of the invariant measure but also, to some extent, of the spectrum, e.g., see [63] and references therein. While most of the numerical work does not track round off errors and hence it is unsatisfactory from the rigorous point of view, some notable exceptions use interval arithmetic and hence have the status of a proof, e.g., [2, 43, 51].

Hence, it is interesting to note that the present approach offers an alternative, possibly much more convenient, route to a numerical computation of the spectrum.

Remark 4.8. We conclude the section with a remark on how the above discussion can provide a numerical scheme to locate eigenvalues. Let $\mathcal{K}g := \mathcal{L}_1(\phi(g)D)$, ϕ being defined in (4.2). Also, let $\{\varphi_i\}_{i=1}^{\infty}$ be a base of $W^{1,1}$ such that, calling Π_N the projection onto $\operatorname{span}\{\varphi_i\}_{i=1}^N$ and $\mathcal{K}_N := \Pi_N \mathcal{K} \Pi_N$, we have

$$\|\mathbb{1} - \Pi_N\|_{W^{1,1} \to L^1} \le C_\sharp N^{-1}.$$

To study the spectrum of $\nu - \mathcal{L}_\star$, $1 > |\nu| > \lambda^{-1}$ when acting on L^1, write

$$\begin{aligned}
\nu - \mathcal{L}_\star &= (\nu - \mathcal{L}_2)\left[\mathbb{1} - (\nu - \mathcal{L}_2)^{-1}\mathcal{K}\right] = (\nu - \mathcal{L}_2)\left[\mathbb{1} - \Pi_N(\nu - \mathcal{L}_2)^{-1}\mathcal{K}\Pi_N + \Delta_N\right] \\
&= (\nu - \mathcal{L}_2)\left[\mathbb{1} - \Pi_N(\nu - \mathcal{L}_2)^{-1}\mathcal{K}\Pi_N\right]\left(\mathbb{1} - \left[\mathbb{1} - \Pi_N(\nu - \mathcal{L}_2)^{-1}\mathcal{K}\Pi_N\right]^{-1}\Delta_N\right), \\
\Delta_N &= (\mathbb{1} - \Pi_N)(\nu - \mathcal{L}_2)^{-1}\mathcal{K} + \Pi_N(\nu - \mathcal{L}_2)^{-1}\mathcal{K}(\mathbb{1} - \Pi_N).
\end{aligned}$$

Note that Lemma 4.5 implies

$$\|\Delta_N\|_{L^1} \le C_\sharp(\lambda^{-1} - |\nu|)^{-1}N^{-1}.$$

Thus ν belongs to the resolvent of $\mathcal{L}_\star$ if

$$\|\left[\mathbb{1} - \Pi_N(\nu - \mathcal{L}_2)^{-1}\mathcal{K}\Pi_N\right]^{-1}\Delta_N\|_{L^1} \le C_\sharp \frac{\|\left[\mathbb{1} - \Pi_N(\nu - \mathcal{L}_2)^{-1}\mathcal{K}\Pi_N\right]^{-1}\|_{L^1}}{(\lambda^{-1} - |\nu|)N} \le 1.$$

Since $\left[\mathbb{1} - \Pi_N(\nu - \mathcal{L}_2)^{-1}\mathcal{K}\Pi_N\right]^{-1}$ is a finite matrix, its norm can be evaluated numerically (and is essentially proportional to the inverse of the determinant), hence it follows that the spectrum of $\mathcal{L}_\star$ is close to the values of ν for which $\Pi_N(\nu - \mathcal{L}_2)^{-1}\mathcal{K}\Pi_N$ has eigenvalue one. This provides a rather quick way to determine rigorously if $\mathcal{L}_1$ has point spectrum outside the spectral radius of $\mathcal{L}_2$, aside from one.

4.0.3 Point spectrum

If we consider class of maps with some special features, it is possible use arguments like the ones put forward in Remark 4.8 to obtain relevant information about the point spectrum without any computer assisted method.

As an example, the next Theorem provides more precise information on the spectrum in a special class of maps. Note that the following approach can be generalised, here we present only the simplest application to illustrate the logic of the argument.

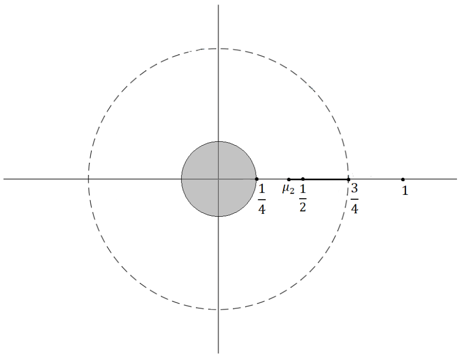
Theorem 4.9. *Let $I := [0,1]$ and $f : I \to I$. Consider the partition $\{(p_i, p_{i+1})\}_{i=1}^N$ to be a partition of a full-measure subset of $[0,1]$ such that for any $1 \le i \le N$, $f([p_i, p_{i+1}]) = [0,1]$, $f \in \mathcal{C}^3([p_i, p_{i+1}], [0,1])$, and $f'(p_i^+) = f'(p_i^-)$, $i \in \{2, \dots, N\}$.*[4] *Also assume that $D \not\equiv 0$ and $D \ge 0$.*[5] *Then,*[6] *for $\mathcal{L}_1 : W^{2,1}([0,1]) \to W^{2,1}([0,1])$*

$$\sigma(\mathcal{L}_1) \subset \left\{ z \in \mathbb{C} \ : \ |z| \le \min\left\{1, \tfrac{2}{f'(1)} - \tfrac{1}{f'(0)}\right\}\right\} \cup \{1\}$$
$$\sigma_{ess}(\mathcal{L}_1) \subset \left\{ z \in \mathbb{C} \ : \ |z| \le \tfrac{1}{f'(1)^2}\right\}.$$

In addition, $\{1\}$ is a simple eigenvalue of $\mathcal{L}_1$ and there exists $\mu_2 < \frac{1}{f'(1)}$ such that $(\mu_2, 1) \cap \sigma(\mathcal{L}_1) = \emptyset$.

Remark 4.10. As an example consider $f(x) = 4x - x^2 \mod 1$. In this case Theorem 4.9 applies with $D = \frac{2}{(4-2x)^2} > 0$, $f'(0) = 4$ and $f'(1) = 2$ and implies that

$$\sigma(\mathcal{L}_1) \subset \left\{ z \in \mathbb{C} \ : \ |z| \le \tfrac{3}{4}\right\} \cup \{1\} \setminus \left\{ z \in \mathbb{C} \ : \ \Re(z) \in \left(\mu_2, \frac{3}{4}\right]\right\}$$
$$\sigma_{\text{ess}}(\mathcal{L}_1) \subset \left\{ z \in \mathbb{C} \ : \ |z| \le \tfrac{1}{4}\right\}.$$



Remark 4.11. Note that if $D \equiv 0$, then $f'(1) = N$ and $\mathcal{L}_1$ has eigenvalue N^{-1} with eigenfunction $g(x) = x - \frac{1}{2}$. Indeed,

$$\mathcal{L}_1 g(x) = \sum_{i=0}^{N-1} N g\left(\frac{x+i}{N}\right) = \sum_{i=0}^{N-1} (x+i) - \frac{N}{2} = N\left(x + \frac{N-1}{2} - \frac{N}{2}\right) = N g(x).$$

By perturbation theory, see [54], it follows that such an eigenvalue survives for small distortion. However, the above theorem implies that,for perturbations satisfying Theorem 4.9, one cannot make it increase more than $\frac{2}{f'(1)} - \frac{1}{f'(0)}$.

[4]By $g(p^{\pm})$ we mean the right and left limit, respectively, of the function g.
[5]Hence there is no need to distinguish between p_i^- and p_i^+, so we will not do it anymore.
[6]Note that the following provides a spectral gap if $f(1) \ge 2$.

Before proving Theorem 4.9 we needs few preliminary lemmata.

In this case it is convenient to define $\psi(g)(x) = \int_0^x g(y)dy$ and

$$\mathcal{L}_+ g = \mathcal{L}_2 g + \mathcal{L}_1(D\psi(g)).$$

Note that $\mathcal{L}_+$ is a positive operator: if $g \geq 0$, then

$$\mathcal{L}_+ g \geq \mathcal{L}_1(D\psi(g)) \geq 0.$$

This facilitate the study of its spectrum. There is an obvious connection it with the operator we are interested in:

$$\mathcal{L}_\star g = \mathcal{L}_+ g - (\mathcal{L}_1 D) \cdot \int_0^1 (1-y)g(y)dy, \tag{4.4}$$

that is $\mathcal{L}_\star$ is a rank one perturbation of $\mathcal{L}_+$.

Before proceeding further we need some information on $\mathcal{L}_+$.

Lemma 4.12. *The spectral radius of $\mathcal{L}_+$, acting on L^1, is $\mu_* := \frac{1}{f'(1)}$. Moreover, μ_* is an eigenvalue of $\mathcal{L}_+'$, acting on L^∞ with eigenvector given by Lebesgue.*

Proof. Note that, for all $g \in L^1$,

$$\begin{aligned}\int_0^1 \mathcal{L}_+ g(y)dy &= \int_0^1 \left[\frac{g(y)}{f'(y)}dy + \left(\frac{1}{f'(y)}\right)'\psi(g)(y)\right]dy = \int_0^1 \left(\frac{\psi(g)}{f'}\right)'(y)\, dy \\ &= \frac{1}{f'(1)}\psi(g)(1) - \frac{1}{f'(0)}\psi(g)(0) = \frac{1}{f'(1)}\int_0^1 g(y)\, dy.\end{aligned} \tag{4.5}$$

Hence, $\frac{1}{f'(1)}$ is an eigenvalue of the dual of $\mathcal{L}_+$ and hence it belongs to the spectrum of $\mathcal{L}_+$. The lemma follows since

$$\int_0^1 |\mathcal{L}_+ g(y)|\, dy \leq \int_0^1 \mathcal{L}_+|g|(y)dy = \frac{1}{f'(1)}\int_0^1 |g(y)|\, dy. \tag{4.6}$$

□

Note that the above Lemma implies that the space $\mathbb{V}_0 = \{h \in L^1 \mid \int_0^1 h = 0\}$ is invariant under $\mathcal{L}_+$. However, this does not give much information on the spectrum. To learn more it is convenient to study the operator $\mathcal{L}_+$ acting on $W^{1,1}$.

Lemma 4.13. *For all $g \in W^{1,1}$ we have*

$$\begin{aligned}&\|\mathcal{L}_+ g\|_{L^1} \leq \mu_*\|g\|_{L^1}\\ &\|\mathcal{L}_+ g\|_{W^{1,1}} \leq \mu_*^2\|g\|_{W^{1,1}} + (3\|D\|_\infty + \|D'\|_{L^1} + \|D^2\|_{L^1} + \mu_*)\|g\|_{L^1}.\end{aligned}$$

Proof. The first inequality follows from (4.6). Next, for each $g \in W^{1,1}$, using again (4.1), we have

$$(\mathcal{L}_+ g)' = \mathcal{L}_3 g' + 3\mathcal{L}_2 Dg + \mathcal{L}_2 D'\psi(g) + \mathcal{L}_1 D^2\psi(g). \tag{4.7}$$

Thus (note that $D \geq 0$ implies that $f'' \leq 0$ and so $f'(0) \geq f(x) \geq f'(1)$),

$$\|(\mathcal{L}_+ g)'\|_{L^1} \leq \mu_*^2\|g'\|_{L^1} + (3\|D\|_\infty + \|D'\|_{L^1} + \|D^2\|_{L^1})\|g\|_{L^1}.$$

The Lemma follows using again (4.6). □

Lemma 4.14. *μ_* is a simple eigenvalue of $\mathcal{L}_+$. In addition, there exists $\mu_1 < \mu_*$ such that $\sigma_{W^{1,1}}(\mathcal{L}_+) \subset \{\mu_*\} \cup \{z \in \mathbb{C} : |z| \leq \mu_1\}$.*

Proof. Lemma 4.13 and Theorem 2.23 imply that the essential spectrum of $\mathcal{L}_+$ is contained in a disk of size μ_*^2. Thus μ_* must be an eigenvalue, let $h_+ \in W^{1,1} \setminus \{0\} \subset \mathcal{C}^0$ be a corresponding eigenvector. Next, suppose that $\mathcal{L}_+ g = \mu_* e^{i\theta} g$, then $\mu_* |g| \leq \mathcal{L}_+ |g|$, but then

$$\int_0^1 \mathcal{L}_+ |g| - \mu_* |g| = 0$$

thus $\mu_* |g| = \mathcal{L}_+ |g|$. Accordingly, we can assume that $h_+ \geq 0$. But then it must be $h_+ > 0$. Indeed, if there exists $\bar{x}$ such that $h_+(\bar{x}) = 0$, then, calling y the maximal element in $f^{-1}(\bar{x})$,

$$0 = \mu_* h_+(\bar{x}) \geq \frac{1}{f'(y)} \left(\frac{1}{f'}\right)'(y) \int_0^y h_+.$$

Hence $h_+(x) = 0$ for all $x \leq y$. Iterating the argument we have that $h_+(x) = 0$ for all $x < 1$, and, by continuity, $h_+ \equiv 0$, contrary to the assumption. Accordingly, if there exists another h such that $\mathcal{L}_+ h = \mu_* h$, then it cannot be zero anywhere otherwise $|h|$, which is also an eigenvalue, would be identically zero. But then there exists $\alpha \in \mathbb{R}$ such that $\alpha h_+ - |h|$ has a zero and hence $h = \alpha h_+$.

Therefore, if $e^{i\theta} \mu_* g = \mathcal{L}_+ g$ then there must exist $\vartheta \in \mathcal{C}^0$ such that $g = e^{i\vartheta} h_+$. It follows

$$\begin{aligned} 0 =& \mu_* h_+ - e^{-i\theta - i\vartheta} \mathcal{L}_+(e^{i\vartheta} h_+) = \mathcal{L}_+ h_+ - e^{-i\theta - i\vartheta} \mathcal{L}_+(e^{i\vartheta} h_+) \\ =& \mathcal{L}_2 \left[1 - e^{-i\theta - i\vartheta \circ f + i\vartheta}\right] h_+ + \mathcal{L}_1 D \left[\psi(g) - e^{-i\theta - i\vartheta \circ f} \psi(e^{i\vartheta} h_+)\right]. \end{aligned}$$

Taking the real part and integrating yields

$$0 = \int_0^1 \frac{1 - \cos[\theta + \vartheta \circ f - \vartheta]}{f'} h_+ + \int_0^1 dx D(x) \int_0^x dy \left[1 - \cos[\theta + \vartheta \circ f(x) - \vartheta(y)\right] h_+(y).$$

Since both terms are positive, the only possibility is $\theta + \vartheta \circ f(x) - \vartheta(y) = k\pi$. This implies that ϑ is constant and hence g is proportional to h_+, hence it must be $\theta = 0$. This proves that μ_+ is the only peripheral eigenvalue and the spectral gap. □

We can now conclude our argument.

Proof of Theorem 4.9. Equation (4.4) and Lemma 4.13 imply the bound on the essential spectral radius.

Since f' is continuous on $[0, 1]$ we know that $\int_0^1 |(\frac{1}{f'(y)})'| dy = 1/f'(1) - 1/f'(0)$. This means that the first statement of the theorem follows from Theorem 4.6 where

$$\tau = \lambda^{-1} + \int_0^1 \left| \left(\frac{1}{f'(y)}\right)' \right| dy = \frac{2}{f'(1)} - \frac{1}{f'(0)}.$$

It remains to show the absence of the eigenvalues in the interval $(\mu_1, 1)$. By equation (4.4) we have that if $\mathcal{L}_\star g = zg$ then

$$(z - \mathcal{L}_+) g = -\mathcal{L}_1 D \int_0^1 (1 - y) g(y). \tag{4.8}$$

Note that, if $|z| > \mu_1$, then the right hand side of the above equation cannot be zero otherwise, by Lemma 4.14, we would have $g = h_+$ and $z = \mu_*$, but then the integral would be strictly positive. By the same argument, since $D \not\equiv 0$, $z \neq \mu_*$. It follows that, possibly after a normalization, for $|z| > \mu_1$ we can write $g = (z - \mathcal{L}_+)^{-1}\mathcal{L}_1 D$, and substituting in (4.8) we have

$$(z - \mathcal{L}_+)^{-1}\mathcal{L}_1 D(x) = -(z - \mathcal{L}_+)^{-1}\mathcal{L}_1 D(x) \int_0^1 (1-y)(z - \mathcal{L}_+)^{-1}\mathcal{L}_1 D(y)dy.$$

Accordingly, if we define

$$\begin{aligned}\Xi(z) :=& 1 + \int_0^1 (1-y)(z - \mathcal{L}_+)^{-1}\mathcal{L}_1 D(y)dy \\ =& 1 + (z-\mu_*)^{-1} \int_0^1 \mathcal{L}_1 D(y)dy - \int_0^1 y(z - \mathcal{L}_+)^{-1}\mathcal{L}_1 D(y)dy \\ =& 1 + (z-\mu_*)^{-1} \left[\frac{1}{f'(1)} - \frac{1}{f'(0)}\right] - \int_0^1 y(z - \mathcal{L}_+)^{-1}\mathcal{L}_1 D(y)dy\end{aligned}$$

we have that z is an eigenvalue of $\mathcal{L}_\star$ if and only if $\Xi(z) = 0$.

Note that, if $z > \mu_*$, then

$$\int_0^1 (1-y)(z - \mathcal{L}_+)^{-1}\mathcal{L}_1 D(y)dy = \sum_{n=0}^{\infty} \int_0^1 (1-y) z^{-n-1}\mathcal{L}_+^n \mathcal{L}_1 D(y)dy > 0.$$

Hence, $\Xi(z) > 1$. On the other hand we can rewrite equation $\Xi(z) = 0$ as

$$\begin{aligned}z &= \mu_* - \left[\frac{1}{f'(1)} - \frac{1}{f'(0)}\right] + \int_0^1 y(z-\mu_*)(z - \mathcal{L}_+)^{-1}\mathcal{L}_1 D(y)dy \\ &=: \mu_* - \left[\frac{1}{f'(1)} - \frac{1}{f'(0)}\right] + \beta(z).\end{aligned} \tag{4.9}$$

Note that β is analytic for $|z| > \mu_1$. By Lemma 4.14 it follows the spectral representation $\mathcal{L}_+ h = \mu_* h_+ \int_0^1 h + Qh$ where for all $\mu > \mu_1$ there exists C_μ such that $\|Q^n\|_{W^{1,1}} \leq C_\mu \mu^n$. Thus

$$\beta(z) = \int_0^1 dy y h_+(y) \int_0^1 \mathcal{L}_1 D(\xi)d\xi + \mathcal{O}(z - \mu_*)$$

which implies

$$\beta(\mu_*) = \left[\frac{1}{f'(1)} - \frac{1}{f'(0)}\right] \int_0^1 y h_+(y)dy < \left[\frac{1}{f'(1)} - \frac{1}{f'(0)}\right].$$

Hence, there exists $\mu_2 < \mu_*$ such that (4.9) has no solution for $z > \mu_2$.[7] □

Different operators

As a last comment on the present approach to the study of the spectrum of $\mathcal{L}_1$, let us remark that it is possible to investigate the commutation relations with different operators. As an example, let us consider the operator $A(h) = h' + \alpha h$ for some function α. Then

$$A\mathcal{L}_1 h = \mathcal{L}_2 h' + \mathcal{L}_1 D h + \mathcal{L}_1(\alpha \circ f h) = \mathcal{L}_2(Ah) + \mathcal{L}_1\left[(D - \frac{\alpha}{f'} + \alpha \circ f)h\right]. \tag{4.10}$$

[7] With some further work one could estimate μ_2, but we believe the above suffices to show how to proceed.

In general, is not obvious what the best choice of α could be. To keep the discussion short let us consider only the special, well known, case in which $\ln f'$ is $\mathcal{C}^1$ cohomologous to a constant.[8] That is, there exists a $\mathcal{C}^1$ function B such that

$$\ln f' + B - B \circ f = c.$$

Then we can choose $\alpha = B'$ and obtain

$$A\mathcal{L}_1 h = \mathcal{L}_2(Ah).$$

Accordingly, if $\mathcal{L}_1 h = \nu h$, $|\nu| \geq \lambda^{-1}$, then $\mathcal{L}_2(Ah) = \nu Ah$, thus $Ah = 0$. This implies that $h = e^{-B}a$, $a \in \mathbb{C}$, hence

$$\mathcal{L}_1 e^{-B} = \mathcal{L}_0 e^{-c-B\circ f} = e^{-c-B}\mathcal{L}_0 1 = e^{-c} N e^{-B}.$$

Integrating yields $e^{-c}N = 1$, hence $\nu = 1$. It follows that

$$\sigma_{\mathcal{C}^1}(\mathcal{L}_1) \subset \{1\} \cup \{z \in \mathbb{C} \,:\, |z| \leq \lambda^{-1}\},$$

hence, as expected, the existence of a large spectral gap.

In the general case, one could try to minimise $D - \frac{\alpha}{f'} + \alpha \circ f$ in order to produce estimates that improve Theorem 4.9.

[8] This happens if f is $\mathcal{C}^2$ conjugated to a map $f_\ell(x) = \ell x \mod 1$, $\ell \in \mathbb{Z}$ with $|\ell| \geq 2$.

Chapter 5

Full branch monotone maps

Up to now we have considered uniformly expanding systems. However much of our arguments were rather general, it is then natural to ask if one can apply the present philosophy also to non-uniformly expanding maps or even maps that expand only in some part of the phase space. We believe the answer to be affirmative and to justify such a belief we discuss one of the simplest possibilities: one dimensional full branch monotone maps (See [64] for full details). Of course, for such more general systems one cannot expect to prove as many results as in the previous section. Yet, some interesting and novel results can be obtained. To illustrate such a fact we will discuss the operator associated to the measures of maximal entropy.

Let $\mathcal{P} = \{I_1, \dots, I_N\}$ be a partition of $[0,1]$ in the sense that the I_i are open disjoint intervals and $\cup_{i=1}^N \overline{I_i} = [0,1]$. Let $f : [0,1] \to [0,1]$ be a map such that $f(I_i) = (0,1)$, $f|_{I_i}$ is invertible and $f|_{I_i} \in \mathcal{C}^1$. Thus each point in $(0,1)$ has exactly N preimages. Suppose that $\Lambda = \|f'\|_\infty < \infty$. We write $\mathbb{M}$ for the set of maps satisfying the above properties.

Remark 5.1. Note that maps in $\mathbb{M}$ can have attracting fixed points or attracting periodic orbits.

Remark 5.2. Note that we ask only $f|_{I_i} \in \mathcal{C}^1$, rather than $f|_{I_i} \in \mathcal{C}^{1+\alpha}$ as is necessary when studying the SRB measure.

Also note that $\Lambda \geq N$ since

$$N = \sum_{i=1}^N |f(I_i)| \leq \sum_{i=1}^N \int_{I_i} |f'(x)|\, dx \leq \Lambda \sum_{i=1}^N |I_i| = \Lambda,$$

where the inequality is strict if f' is not constant.

Note that, identifying 0 and 1, we could see $f \in \mathbb{M}$ as a piecewise monotone map from the circle to itself.

We want to investigate the mixing rate for the measure of maximal entropy. We start recalling some well known facts (see [21] for a review).

Lemma 5.3 ([65, Theorem 1]). *For $f \in \mathbb{M}$ holds the variational principle*

$$h_{top} = \ln N = \sup_{\mu \in \mathcal{M}} h_\mu(f)$$

where $\mathcal{M}$ is the set of invariant measures of f and $h_\mu(f)$ is the Kolmogorov-Sinai entropy.

Our next goal is to construct a measure μ_{BM} of maximal entropy by using the transfer operator $\mathcal{L}_0$ introduced in the previous sections.

We start by noticing that, for each $h \in \mathcal{C}^1$,

$$\begin{aligned} \|\mathcal{L}_0 h\|_{L^1} &\leq \int_0^1 \mathcal{L}_1 |f'h| = \int |f'||h| \leq \Lambda \|h\|_{L^1}, \\ \|(\mathcal{L}_0 h)'\|_{L^1} &= \|\mathcal{L}_1 h'\|_{L^1} \leq \|h'\|_{L^1}. \end{aligned} \tag{5.1}$$

We have thus the Lasota-Yorke inequality

$$\begin{aligned} \|\mathcal{L}_0 h\|_{L^1} &\leq \Lambda \|h\|_{L^1}, \\ \|(\mathcal{L}_0 h)\|_{W^{1,1}} &\leq \|h\|_{W^{1,1}} + \Lambda \|h\|_{L^1}. \end{aligned} \tag{5.2}$$

From (5.2) and Hennion's Theorem 2.23 it follows that the spectral radius, on $W^{1,1}$, of $\mathcal{L}_0$ is bounded by Λ while the essential spectral radius is bounded by one.

Theorem 5.4. *The operator $\mathcal{L}_0$ when acting on $W^{1,1}$ has the spectral decomposition $\mathcal{L}_0 h = N \cdot \mu_{\text{BM}}(h) + Q(h)$ where $Q1 = 0$, $\mu_{\text{BM}}(Q(h)) = 0$, for all $h \in W^{1,1}$, and $\sigma_{W^{1,1}}(Q) \subset \{z \in \mathbb{C} : |z| \leq 1\}$.*

Proof. We know that if $\nu \in \sigma(\mathcal{L}_0)$ and $|\nu| > 1$, then ν is point spectrum. That is there exist $h \in W^{1,1}$ such that $\mathcal{L}_0 h = \nu h$. But then, differentiating, we have $\mathcal{L}_1 h' = \nu h'$ where $h' \in L^1$. However, $\mathcal{L}_1$ is a contraction on L^1, hence it must be either $|\nu| \leq 1$, contrary to the hypothesis, or $h' = 0$. The latter implies that h is constant, hence, we can always normalise it so that $h = 1$. On the other hand $\mathcal{L}_0 1(x) = \sum_{y \in f^{-1}(x)} 1 = N$. Hence $\nu = N$ and has geometric multiplicity one. If the geometric multiplicity is not one, then there must exists $h \in W^{1,1}$ such that $\mathcal{L}_0 h = Nh + c$ for some constant c. But then, differentiating, $\mathcal{L}_1 h' = Nh'$, so h must be constant again, hence the maximal eigenvalue is simple.

It thus follows that $\mathcal{L}_0 = N1 \otimes \mu + Q$ where Q has spectral radius smaller or equal one, $Q1 = 0$, $\mu(Qh) = 0$ for all $h \in W^{1,1}$, $\mu(1) = 1$, and μ belongs to the dual of $W^{1,1}$. It remains to prove that μ is a measure and, indeed, a measure of maximal entropy μ_{BM}.

Note that, for each $h \in W^{1,1}$,

$$|\mu(h)| = \lim_{n\to\infty} \left| \int_0^1 N^{-n} \mathcal{L}_0^n h \right| \leq \lim_{n\to\infty} \|h\|_\infty \int_0^1 N^{-n} \mathcal{L}_0^n 1 = \|h\|_\infty.$$

Thus μ is a measure. In addition, for each $h \in \mathcal{C}^1$ such that $h \geq 0$, we have

$$\mu(h) = \lim_{n\to\infty} \int_0^1 N^{-n} \mathcal{L}_0^n h \geq 0$$

thus μ is a positive measure and, since it is normalized, it is a probability measure. Next, note that

$$\mu(\mathcal{L}_0 h) = \lim_{n\to\infty} \int_0^1 N^{-n} \mathcal{L}_0^{n+1} h = N \lim_{n\to\infty} \int_0^1 N^{-n} \mathcal{L}_0^n h = N \cdot \mu(h).$$

It follows

$$\mu(h \circ f) = N^{-1} \mu(\mathcal{L}_0 h \circ f) = N^{-1} \mu(h \mathcal{L}_0 1) = \mu(h).$$

That is μ is an invariant measure. In addition, by the above considerations, $([0,1], f, \mu)$ is ergodic.

The proof is concluded if we show $h_\mu(f) \geq h_{\text{top}}$. Let $\mathcal{P}_n$ denote the n^{th}-refinement of the partition $\mathcal{P}$. Let $p \in \mathcal{P}_n$ and $p_-, p_+ \in \mathcal{P}_n$ be the elements on the left and the right of p, respectively, if they exist. Let $J = p_- \cup p \cup p_+$. Let $h \in \mathcal{C}^1(\mathbb{R}, [0,1])$ be supported in J and such that $h|_p = 1$. Then $\mathcal{L}_0^n h(x) \leq 3$ and

$$\mu(p) \leq \mu(h) = \lim_{m\to\infty} \int_0^1 N^{-m-n}\mathcal{L}_0^{m+n} h \leq \lim_{m\to\infty} 3 \int_0^1 N^{-m-n}\mathcal{L}_0^m 1$$
$$= 3N^{-n}\mu(1) = 3N^{-n}.$$

Accordingly, calling $p_n(x)$ the element of $\mathcal{P}_n$ which contains x, the Shannon-McMillan-Breiman Theorem (e.g., see [71, Section 6.2, Theorem 2.3]) states that for μ almost every point

$$h_\nu(f) \geq h_\mu(\mathcal{P}, f) = \lim_{n\to\infty} -\frac{1}{n} \ln \mu(I_n(x)) \geq \lim_{n\to\infty} \ln N^{1-1/n} = \ln N,$$

which concludes the proof by Lemma 5.3. □

Remark 5.5. We do not know if μ_{BM} is unique in this case, we have just constructed a μ_{BM}. Look at next subsection to see a case where it is easy to prove that the measure of maximal entropy is unique.

Remark 5.6. The monotone interval maps and transfer operators studied in this section fit into the framework considered in [5]. In the reference the essential spectral radius (as operators acting on BV) is obtained and consequently a spectral decomposition. Here we show that the spectral gap is large for the operator associated to the measure of maximal entropy.

We have finally the announced mixing rate estimate

Corollary 5.7. *For any $\nu > \frac{1}{N}$ there exists $C_\nu > 0$ such that, for each $h \in W^{1,1}$ and $\varphi \in L^1(\mu_{\text{BM}})$*

$$\left| \int h\ \varphi \circ f^n\ d\mu_{\text{BM}} - \int h\ d\mu_{\text{BM}} \int \varphi \circ f^n\ d\mu_{\text{BM}} \right| \leq C_\nu \nu^n \|h\|_{W^{1,1}} \|\varphi\|_{L^1(\mu_{\text{BM}})}.$$

Proof. We start assuming that $\varphi \in \mathcal{C}^1$. Then, using Theorem 5.4,

$$\left| \int h\ \varphi \circ f^n\ d\mu_{\text{BM}} - \int h\ d\mu_{\text{BM}} \int \varphi\ d\mu_{\text{BM}} \right|$$
$$= \left| \lim_{m\to\infty} \int_0^1 N^{-m}(\mathcal{L}_0^m h\ \varphi \circ f^n)(x)\ dx - \int h d\mu_{\text{BM}} \int \varphi\ d\mu_{\text{BM}} \right|$$
$$= \left| \lim_{m\to\infty} \int_0^1 [N^{-m+n}\mathcal{L}_0^{m-n}\varphi N^{-n}\mathcal{L}_0^n h](x)\ dx - \int h d\mu_{\text{BM}} \int \varphi\ d\mu_{\text{BM}} \right|$$
$$= \left| \int \varphi N^{-n}\mathcal{L}_0^n h\ d\mu_{\text{BM}} - \int h\ d\mu_{\text{BM}} \int \varphi\ d\mu_{\text{BM}} \right|$$
$$= \left| \int \varphi N^{-n} Q^n h\ d\mu_{\text{BM}} \right| \leq C_\nu \nu^n \|h\|_{W^{1,1}} \int |\varphi|\ d\mu_{\text{BM}}.$$

The corollary follows by a simple approximation argument. □

5.0.1 Non-uniformly expanding maps

Let $\mathcal{A} \subset \mathbb{M}$ the set of maps such that $f' \geq 1$, $f' = 1$ at finitely many points and $\Lambda = \|f'\|_\infty < \infty$.

This class of maps includes the well known Manneville–Pomeau map [72, 62].

Remark 5.8. In [24] non-uniformly expanding systems are studied and the existence of a spectral gap (and hence decay of correlations) is proven for a class of equilibrium states which includes the measure of maximal entropy. The approach in [24] is based on Hilbert metrics and, although not stated explicitly, it provides a poor estimate of the spectral gap (see Remark 4.7 for similar considerations) whereas our present approach provides an explicit and close to optimal estimate.

Here, we limit ourselves to the one dimensional case to present the idea in its simpler form. It is likely that similar results can be obtained for more general non-uniformly expanding maps, e.g., the higher dimensional examples treated in [24].

In this case we can prove that the measure of maximal entropy is unique.

Lemma 5.9. *Any map $f \in \mathcal{A}$ is expansive.*

Proof. Let $\kappa = \min_{I\in\mathcal{P}} |I|$. For each $\delta > 0$ let $\mathcal{I}_\delta = \{[a,b] \subset [0,1] \;:\; [a,b] \subset \bar{I}, I \in \mathcal{P}\;; |b-a| \geq \delta\}$ and, for each $[a,b] \in \mathcal{I}_\delta$, define $\varphi(a,b) := \frac{1}{|b-a|}\int_a^b f'(\xi)d\xi$. Note that, by hypothesis, $\varphi(a,b) > 1$, and since it depends continuously from a, b (which vary in a compact set) there must be $\tau_\delta > 1$ such $\varphi(a,b) \geq \tau_\delta$. Accordingly, $f^n(x)$ and $f^n(y))$ always belong to the same partition element we have $|f^n(x) - f^n(y))| \leq \kappa$ for all $n \in \mathbb{N}$ which is possible only for $x = y$. On the other hand, if for some $f^n(x)$ and $f^n(y)$ belong to two different partition element, then either $|f^n(x) - f^n(y)| \geq \kappa$ or they belong to contiguous elements of $\mathcal{P}$. In such a case it is easy to see that there exists δ such that either $|f^n(x) - f^n(y)| \geq \delta$ or $|f^{n+1}(x) - f^{n+1}(y)| \geq \delta$, hence the expansivity. □

Lemma 5.10. *For $f \in \mathcal{A}$ the measure of maximal entropy μ_{BM} is unique.*

Proof. Since the map is expansive, there exists a map $\Phi : [0,1] \to \{1,\ldots,d\}^{\mathbb{N}} =: \Sigma$ which is well defined and invertible, apart from countably many points, that conjugates f with the full shift σ. Hence, Φ induces a measurable isomorphism for each non-atomic measure. On the other hand for (Σ, σ) holds the variational principle, hence the sup of the metric entropies is the topological entropy, which is $\ln N$, and there exists a unique measure of maximal entropy. Since atomic measures have zero entropy, and since the entropy is an affine function of the measures, it follows that the sup on the measure entropies is achieved on non-atomic measures. Thus, via the isomorphism Φ and since the entropy is an invariant for measure-preserving conjugacy, it follows that measure of maximal entropy for f is unique. □

Chapter 6

Hyperbolic maps

For hyperbolic, or partially hyperbolic maps the situation is less clear than in the expanding case and much more remains to be understood. Yet, the ghost of a general theory seems to be present. Let us start with the simplest possible case: linear maps.

In this case it is possible to study the problem using Fourier series (see [59]), however it is interesting to develop an alternative approach that does not rely on the algebraic structure of the map and thus has the potential to be applicable in greater generality.

6.0.1 Automorphisms of the torus

Here we consider a linear map $f : \mathbb{T}^n \to \mathbb{T}^n$ defined by $f(x) = Ax \mod 1$ where $A \in SL(d, \mathbb{Z})$, i.e., a matrix with integer coefficient and $\det A = 1$. Let us call E^u the unstable subspace, E^s the stable one and E^c the central one.

Note that, by hypothesis f preserves the volume, thus the volume is the SRB measure. We are interested in its statistical properties, hence in the transfer operator

$$\mathcal{L}h = h \circ f^{-1}.$$

Next we introduce a norm. Let $\{v_i^s\}$, $\|v_i^s\| = 1$, be a basis of E^s and $\{v_i^u\}$, $\|v_i^u\| = 1$, be a basis of E^u and define $\partial_i^{s/u} h = \langle v_i^{s/u}, \nabla h\rangle$,

$$\begin{aligned} |\varphi|_q^s &= \sup_{0\leq k\leq q} \sup_{i_1,\dots,i_k} \|\partial_{i_1}^s \cdots \partial_{i_k}^s \varphi\|_\infty \\ \|h\|_{p,q} &= \sum_{0\leq k\leq p} \sup_{i_1,\dots,i_k} \sup_{|\varphi|_{k+q}^s\leq 1} \int_{\mathbb{T}^n} \varphi \partial_{i_1}^u \cdots \partial_{i_k}^u h. \end{aligned} \tag{6.1}$$

We call $\mathcal{B}^{p,q}$ the completion of $\mathcal{C}^\infty$ with respect to the norms $\|\cdot\|_{p,q}$.

In the following we assume $E^u \neq \{0\}$. In addition, to simplify the exposition, we assume that A has no Jordan blocks. The general case can be treated with a slight sophistication of the following arguments. We can thus choose the v_i^u such that $Av_i^u = \lambda_i v_i^u$, with $\lambda_i \geq \lambda > 1$. Also let λ be such that $\|A|_{E^s}\| \leq \lambda^{-1}$.

Remark 6.1. The above norms are inspired by [3]. They are one of the many possible constructions of anisotropic Banach spaces adapted to hyperbolic maps or flows, see [7] for an extensive discussion. Given the linear structure of the invariant foliations, the norms (6.1) turn out to be especially convenient and simple to deal with, hence allowing a completely self-contained discussion. In the next section, we will use instead the norms defined in [46] in order to avoid having to redevelop the all theory (e.g. the Lasota-Yorke inequality) in the style of [3], which would certainly be possible.

The following is the equivalent of [3, Proposition 3.2].

Proposition 6.2. *There exist $D, B_q > 0$ such that, for each $p, q \in \mathbb{N}$, we have*

$$\|\mathcal{L}h\|_{p,q} \le \|h\|_{p,q}$$
$$\|\mathcal{L}h\|_{p,q} \le D\lambda^{-\min\{p,q\}}\|h\|_{p,q} + B_q\|h\|_{p-1,q+1}.$$

Proof. Since $\langle v, \nabla(h \circ f^{-1})\rangle = \langle Df^{-1}v, \nabla h\rangle \circ f^{-1}$. We have

$$\int_{\mathbb{T}^n} \varphi\partial^u_{i_1} \dots \partial^u_{i_k}\mathcal{L}h = \prod_{j=1}^{k} \lambda_{i_j}^{-1} \int_{\mathbb{T}^n} \varphi \circ f \partial^u_{i_1} \cdots \partial^u_{i_k} h.$$

Since $|\varphi \circ f|_{k+q} \le |\varphi|_{k+q}$, the first inequality follows.

Next, note that

$$\|h\|_{p,q} = \sup_{i_1,\dots,i_p} \sup_{|\varphi|^s_{p+q}\le 1} \int_{\mathbb{T}^n} \varphi\partial^u_{i_1} \cdots \partial^u_{i_p} h + \|h\|_{p-1,q}.$$

Thus, by the above computations,

$$\|\mathcal{L}h\|_{p,q} \le \lambda^{-p}\|h\|_{p,q} + \|\mathcal{L}h\|_{p-1,q}.$$

It thus suffices to consider the case $k < p$. If $|\varphi|_{k+q} \le 1$, then, for each $\varepsilon > 0$, let φ_ε be such that $|\varphi - \varphi_\varepsilon|^s_{k+q-1} \le \varepsilon$, $|\varphi - \varphi_\varepsilon|^s_{k+q} \le 2$ and $|\varphi_\varepsilon|^s_{k+q+1} \le C\varepsilon^{-1}$, for some fixed constant $C > 2$.[1]

$$\begin{aligned}\left|\int_{\mathbb{T}^n} \varphi\partial^u_{i_1} \cdots \partial^u_{i_k} h\right| &\le \prod_{j=1}^{k} \lambda_{i_j}^{-1} \left\{\left|\int_{\mathbb{T}^n} (\varphi - \varphi_\varepsilon) \circ f\partial^u_{i_1} \cdots \partial^u_{i_k} h\right| + C\varepsilon^{-1}\|h\|_{k,q+1}\right\}\\ &\le \prod_{j=1}^{k} \lambda_{i_j}^{-1} \left\{\max\{\varepsilon, 2\lambda^{-(k+q)}\}\|h\|_{k,q} + C\varepsilon^{-1}\|h\|_{k,q+1}\right\}\\ &\le 2\lambda^{-(2k+q)}\|h\|_{k,q} + C\lambda^q\|h\|_{k,q+1}\end{aligned}$$

where, in the last line, we have chosen $\varepsilon = 2\lambda^{-k-q}$. Accordingly,

$$\|\mathcal{L}h\|_{p,q} \le (\lambda^{-p} + 2\lambda^{-q})\|h\|_{p,q} + C\lambda^q\|h\|_{k,q+1}.$$

The Proposition readily follows. □

Remark 6.3. Note that Proposition 6.2 implies that the spectral radius of $\mathcal{L}$ when acting on any space $\mathcal{B}^{p,q}$ is bounded by one. On the other hand, since $\mathcal{L}1 = 1$, the spectral radius must be exactly one.

The following is the equivalent of [3, Lemma 4.1], although the proof follows a different path, easier in this particular case.

Lemma 6.4. *If $E^c = \{0\}$, then, for each $p, q \in \mathbb{N}$, $\{h \in \mathcal{B}^{p,q} : \|h\|_{p,q} \le 1\}$ is relatively compact in $\mathcal{B}^{p-1,q+1}$.*

[1] Such a function can be constructed by convolving with a mollifier in the space E^s.

Proof. Let $d_s = \dim(E^s)$ and $d_u = \dim(E^u)$. By hypothesis $d = d_s + d_u$.

Let $U : \mathbb{R}^{d_s} \to \mathbb{R}^{d_u}$ such that $\{(v, Uv)\}_{v\in\mathbb{R}^{d_s}} = E^s$ and $V : \mathbb{R}^{d_u} \to \mathbb{R}^{d_s}$ such that $\{(Vv, v)\}_{v\in\mathbb{R}^{d_u}} = E^u$.[2]

Finally, consider mollifiers $j^{s/u}(x) = \varepsilon^{-d_{s/u}} j^{s/u}(\varepsilon^{-1}x)$, where $j^{s/u} \in \mathcal{C}^\infty(\mathbb{R}^{d_{s/u}}, \mathbb{R}_+)$ such that $\text{supp} j^{s/u} \subset \{\|x\| \le 1\}$ and $\int_{\mathbb{R}^{d_{s/u}}} j^{s/u}(x)dx = 1$. Then, for each $|\varphi|_{p+q} \le 1$

$$
\begin{aligned}
&\int_{\mathbb{T}^d} dx \varphi \partial^u_{i_1} \cdots \partial^u_{i_{p-1}} h = \int_{\mathbb{T}^d} dx \int_{\mathbb{R}^{d_s}} dv \varphi(x+v) j^s_\varepsilon(v) \partial^u_{i_1} \cdots \partial^u_{i_{p-1}} h(x) + \mathcal{O}(\varepsilon \|h\|_{p,q}) \\
&= \int_{\mathbb{T}^d} dx \int_{\mathbb{R}^{d_s}} dv \int_{\mathbb{R}^{d_s}} dw \varphi(x+v-w) j^s_\varepsilon(v) j^u_\varepsilon(w) \partial^u_{i_1} \cdots \partial^u_{i_{p-1}} h(x) + \mathcal{O}(\varepsilon \|h\|_{p,q}) \\
&= \int_{\mathbb{T}^d} dx\, \boldsymbol{\varphi}_\varepsilon(x) \partial^u_{i_1} \cdots \partial^u_{i_{p-1}} h(x) + \mathcal{O}(\varepsilon \|h\|_{p,q}).
\end{aligned}
$$

Note that $\|\boldsymbol{\varphi}_\varepsilon\|_{\mathcal{C}^{p+q+1}} \le C\varepsilon^{-p-1}$ and hence for each ε there is a set $\{\phi_i\}_{i=1}^{N_\varepsilon} \subset \mathcal{C}^{p+q}$ such that, for all φ we have $\|\boldsymbol{\varphi}_\varepsilon - \phi_i\|_{\mathcal{C}^{p+q}} \le \varepsilon$ for some ϕ_i. It follows that, for each ε,

$$
\|h\|_{p-1,q+1} \le C_\sharp \varepsilon \|h\|_{p,q} + \sup_{i \le N_\varepsilon} \left| \int_{\mathbb{T}^d} \phi_i h \right|.
$$

From the above the wanted compactness follows by a standard diagonalization argument. □

We can now define the operators $\mathcal{D}_i h = \partial^s_i h$. Then

$$
\mathcal{D}_i \mathcal{L} f = \langle Df^{-1} v_i, \nabla h \circ f^{-1} \rangle = \lambda_i \langle v_i, h \rangle \circ f^{-1} = \lambda_i \mathcal{L} \mathcal{D}_i h. \tag{6.2}
$$

The usefulness of these operators rests in the following Lemma. This is the only place in which we use Fourier series, however the result follows essentially from the accessibility property although with a more cumbersome proof.

Lemma 6.5. *If $h \in \mathcal{B}^{p,q}$, $p \ge 1$ and $\mathcal{D}_i h = 0$, for all i, then h is constant. In addition, the $\mathcal{D}_i$ are bounded operators from $\mathcal{B}^{p-1,q+1}$ to $\mathcal{B}^{p,q}$.*

Proof. Recall that Katznelson Lemma [53] (applied to $(A^{-1})^*$) implies that there exists $C_0 > 0$ such that, for each $z \in \mathbb{Z}^n \setminus \{0\}$,

$$
\text{dist}(z, (E^s \oplus E^c)^\perp) \ge C_0 \|z\|^{-n}, \tag{6.3}
$$

Let $\hat{h}_k \ne 0$, $k \ne 0$, be a Fourier coefficients of h. Then $\widehat{\mathcal{D}_j h}_k = i\langle v^s_j, k\rangle \hat{h}_k$. But if $\langle v^s_j, k\rangle = 0$ for all j, then $k \perp E^s$, contradicting (6.3). Thus h must be constant. The fact that the $\mathcal{D}_j$ are bounded operators from $\mathcal{B}^{p-1,q+1}$ to $\mathcal{B}^{p,q}$ is a direct consequence of the definition of the norms in (6.1) and integration by parts. □

We are now ready to draw our conclusions.

Lemma 6.6. *If $E^c = \{0\}$, then for each $\varepsilon > 0$ and p, q are large enough we have $\sigma_{\mathcal{B}^{p,q}}(\mathcal{L}) \subset \{1\} \cup \{z \in \mathbb{C} : |z| \le \varepsilon\}$.*

Proof. By Proposition 6.2 and Lemma 6.4, together with Theorem 2.23 we have that the spectrum in the considered region is only point spectrum provided $\lambda^{-\min\{p,q\}} < \varepsilon$. We thus require p, q to be such that $\lambda^{-\min\{p,q\}} < \varepsilon$.

Next, suppose that $\mathcal{L}h = \nu h$ with $|\nu| > \varepsilon$, then, for all j, (6.2) implies

$$
\mathcal{L}(\mathcal{D}^q_j h) = \lambda_j^{-q} \nu (\mathcal{D}^q_j h).
$$

[2] We can always choose coordinates in which this is possible.

But since $|\lambda_j^q \nu| > 1$ it cannot be an eigenvalue of $\mathcal{L}$, thus it must be $\mathcal{D}_j^q h = 0$. But, since integrating by parts yields

$$\int_{\mathbb{T}^d} \mathcal{D}_l^q h = 0,$$

for all $0 < l \leq j$, Lemma 6.5 implies that h must be a constant, which, in turn, implies $\nu = 1$. □

Remark 6.7. Lemma 6.6, by the usual arguments, implies that $\mathcal{C}^\infty$ observables have a super-exponential decay of correlations.

Remark 6.8. It seems reasonable to expect that a similar result, albeit possibly by using a different Banach space, should hold also if E^c is not trivial. However, the proof of Lemma 6.4 fails in this case so it is unclear how to obtain the needed compactness. Hence, at present, it is not clear how to apply this strategy to partially hyperbolic systems, even in the simplest case.

The nonlinear case is much more subtle even in the Anosov setting. The obvious idea would be to consider an unstable vector field w and the operator $\mathcal{D} = \langle w, \nabla h \rangle$. Unfortunately, in general, unstable vector fields are only Hölder. Hence, it is not clear if $\mathcal{D}$ is a well defined bounded operator from $\mathcal{B}^{p,q}$ to $\mathcal{B}^{p-1,q+1}$. To solve this problem one should probably use different banach spaces in the spirit of (for some appropriate version of such spaces such as the ones introduced in [45], see [79] for some recent progress along these lines).

Indeed, on the one hand w is smooth along unstable manifolds, on the other hand in the stable direction is only Hölder so its derivatives must be regarded as distribution, like h, and multiplication of distributions is a rather touchy business. So the situation, although not hopeless, is rather unclear.

Such issue needs further thought. Here we limit ourselves to explore an interesting alternative: considering the external derivative d as the appropriate differential operator. This simple change of perspective yields interesting results since it seems to provide a connection with the topology of the manifold. At least, this is the situation in the following where we discuss only the simplest case: two dimensional Anosov maps.

6.0.2 Anosov map on two dimensional manifolds

Let M be a smooth two dimensional compact and connected Riemannian manifold and $f \in \text{Diff}^\infty(M, M)$, be a transitive Anosov map. In other words, there exists $\lambda > 1$ and two continuous strictly invariant cone fields $\mathcal{C}^s, \mathcal{C}^u$ such that, for all $x \in M$,

$$\|d_x f v\| \geq \lambda \|v\| \quad \forall v \in \mathcal{C}^u(x)$$
$$\|d_x f^{-1} v\| \geq \lambda \|v\| \quad \forall v \in \mathcal{C}^s(x).$$

Remark 6.9. According to the Franks-Newhouse Theorem [42, 66], every Anosov diffeomorphism of a two-dimensional compact Riemannian manifold is topologically conjugate to a hyperbolic toral automorphism. Hence, in our case, M must be homeomorphic to $\mathbb{T}^2$. Note however that in the following the smoothness of the map plays a fundamental role, hence one cannot in general reduce the discussion to the case $\mathbb{T}^2 = \mathbb{R}^2 \backslash \mathbb{Z}^2$. It is thus convenient to argue considering M a general two dimensional manifold. This has also the advantage to emphasise the possibility of a higher dimensional extension. Indeed, we will use the Franks-Newhouse Theorem only at the end of the argument (Lemma 6.20), to characterise the cohomology groups.

In analogy with the previous sections, we will obtain results on the mixing properties of the measure of maximal entropy $\mu_{\rm BM}$.

Theorem 6.10. *The exists $r \in \mathbb{N}$, $C > 0$ and $\kappa \in (0,1)$ such that for all $g, h \in \mathcal{C}^\infty$ and $n \in \mathbb{N}$ we have*

$$\left| \int_M g \circ f^n h d\mu_{\rm BM} - \int_M g d\mu_{\rm BM} \int_M h d\mu_{\rm BM} \right| \le C \|g\|_{\mathcal{C}^r} \|h\|_{\mathcal{C}^r} e^{-h_{top} n} \kappa^n.$$

This result is a corollary of the much more precise Theorem 6.11 and it is proven in section 6.0.6. To state Theorem 6.11 we need to first introduce several objects.

6.0.3 The operators

The operator associated to the SRB measure is simply (e.g., see [44])

$$\mathcal{L}h(x) = (\det D_{f^{-1}(x)} f)^{-1} h \circ f^{-1}(x).$$

However, in the present context the interesting object to study seems to be the action of forms, or rather *currents*.[3] Recall that the pullback on a differential form ω by a map g is defined as

$$(g^*\omega)_x(v_1, v_2) = \omega_{g(x)}(d_x g(v_1), d_x g(v_2)).$$

If g is a diffeomorphism we can define the pushforward as $g_*\omega = (g^{-1})^*\omega$. It is then natural to define the action of the dynamics on forms as the pushforward f_*.

Let ω_0 be the Riemannian volume. Then any two form can be written as $\omega = h\omega_0$ for some function h. Then

$$\begin{aligned}
[f_*\omega(v_1, v_2)](x) &= h \circ f^{-1}(x) \omega_0((d_x f^{-1}(v_1), d_x f^{-1}(v_2)) \\
&= h \circ f^{-1}(x) \det(D_{f^{-1}(x)} f)^{-1} \omega_0(v_1, v_2)(x) \\
&= [\mathcal{L}h \cdot \omega_0](v_1, v_2)(x),
\end{aligned} \tag{6.4}$$

That is, the operator $\mathcal{L}$ is equivalent to the pushforward on two forms.

Recall that

$$d(f_* h) = f_* dh. \tag{6.5}$$

where, if h is a zero form, then $f_* dh(x) = \left[D_x f^{-1}\right]^T (\nabla h) \circ f^{-1}(x)$.

The scalar product in T^*M is canonically defined by using the canonical duality $\pi : T^*M \to T_*M$ defined by $\omega(v) = \langle \pi(\omega), v\rangle$, for all $v \in T_*M$. That is,

$$\langle \omega_1, \omega_2 \rangle = \langle \pi(\omega_1), \pi(\omega_2)\rangle = \omega_1(\pi(\omega_2)). \tag{6.6}$$

For each $x \in M$ and $v_1, v_2, w_1, w_2 \in T^*_x M$ we define

$$\langle v_1 \wedge v_2, w_1 \wedge w_2 \rangle = \det(\langle v_i, w_j \rangle). \tag{6.7}$$

Assuming bilinearity, the above formula defines uniquely a scalar product among 2-forms. Also, we define a duality from ℓ to $2-\ell$ forms via (see [46, Appendix A] for more details)

$$\langle v, w \rangle \omega_0 = (-1)^{\ell(2-\ell)} v \wedge *w = (-1)^{\ell(2-\ell)} w \wedge *v = *v \wedge w. \tag{6.8}$$

Since such a formula must hold for all ℓ-forms, the $(2-\ell)$-forms $*w, *v$ are uniquely defined. The operator "$*$" is the so called *Hodge operator*.

[3] The idea that currents are a relevant object to study in the context of the statistical properties of dynamical systems goes back, at least, to [76]. See, for example, [75] and [4] for further use of k-forms in the dynamical systems context.

6.0.4 The Banach spaces and the main result

The operators f_* have been studied for flows in [46] using appropriate Banach spaces. We use the same notation and almost the same Banach spaces defined in [46, Section 3]. However, since here we consider maps rather than flows, we do not have the requirement that the forms be null in the flow direction (see [46, equation (3.5)]).

For any $r \in \mathbb{N}$, we assume that there exists $\delta_0 > 0$ such that, for each $\delta \in (0, \delta_0)$ and $\rho \in (0, 4)$, there exists an atlas $\{(U_\alpha, \Theta_\alpha)\}_{\alpha \in A}$, where A is a finite set, such that[4]

$$\begin{cases} \Theta(U_\alpha) = B_2(0, 30\delta\sqrt{1+\rho^2}), \\ \cup_\alpha \Theta_\alpha^{-1}(B_2(0, 2\delta)) = M, \\ \|(\Theta_\alpha)_*\|_\infty + \|(\Theta_\alpha^{-1})_*\|_\infty \leq 2; \quad \|\Theta_\alpha \circ \Theta_\beta^{-1}\|_{C^r} \leq 2. \end{cases} \tag{6.9}$$

Fix $L_0 > 0$. For any $L > L_0$, let us define

$$\mathcal{F}_r(\rho, L) := \{F : B_1(0, 6\delta) \to \mathbb{R} : F(0) = 0;$$

$$\|DF\|_{C^0(B_1(0,6\delta))} \leq \rho; \|F\|_{C^r(B_1(0,6\delta))} \leq L\}.$$

Where the C^r is defined as usual, e.g. see [46, equation (3.6)]. For each $F \in \mathcal{F}_r(\rho, L)$, $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^1$, let $G_{x,F}(\xi) : B_1(0, 6\delta) \to \mathbb{R}^2$ be defined by $G_{x,F}(\xi) := x + (\xi, F(\xi))$. Let us also define $\tilde{\Sigma}(\rho, L) := \{G_{x,F} : x \in B_1(0, 2\delta), F \in \mathcal{F}_r(\rho, L)\}$. For each $\alpha \in A$ and $G \in \tilde{\Sigma}(\rho, L)$, we define the leaf

$$W_{\alpha,G} := \{\Theta_\alpha^{-1} \circ G(\xi)\}_{\xi \in B_1(0,3\delta)}.$$

For each $\alpha \in A$, $G \in \tilde{\Sigma}(\rho, L)$, note that $W_{\alpha,G} \subset \hat{U}_\alpha := \Theta_\alpha^{-1}(B_d(0, 6\delta\sqrt{1+\rho^2})) \subseteq U_\alpha$. Finally, we define $\Sigma_\alpha = \cup_{G \in \tilde{\Sigma}(\rho,L)} W_{\alpha,G}$.
Given a curve $W_{\alpha,G} \in \Sigma_\alpha$, we consider $\Gamma_c^{\ell,s}(\alpha, G)$ as the C^s sections of the fiber bundle on $W_{\alpha,G}$ with fibers in $\wedge^\ell(T^*M)$, as defined in [46, equation (3.8)].
Following [46, Section 3] let $V^s(\alpha, G)$ be the set of uniformly $C^s(U_{\alpha,G})$ vector fields, where $U_{\alpha,G}$ is any open set such that $U_\alpha \supset U_{\alpha,G} \supset W_{\alpha,G}$.

Let ω_{vol} be the d_s volume form induced on $W_{\alpha,G}$ by the push-forward of Lebesgue measure via the chart Θ_α^{-1}. Write L_ν for the Lie derivative along a vector field ν. For all $\alpha \in A, G \in \Sigma_\alpha, g \in \Gamma_c^{l,0}(\alpha, G), \bar{v}^p = (v_1, \cdots, v_p) \in V^s(\alpha, G)^p$, let us define the functionals $J_{\alpha,G,g,\bar{v}^p} : C^p \to \mathbb{C}$ by

$$J_{\alpha,G,g,\bar{v}^p}(h) = \int_{W_{\alpha,G}} \langle g, L_{v_1} \cdots L_{v_p} h \rangle \omega_{vol}.$$

Next, for all $p \in \mathbb{N}$, $q \in \mathbb{R}_+$, $p + q < r - 1$, $l \in \{0, \cdots, d\}$, let

$$\mathbb{U}_{\rho,L,p,q,\ell} = \{J_{\alpha,G,g,\bar{\nu}^p} | \alpha \in A, G \in \Sigma_\alpha(\rho, L), g \in \Gamma_c^{\ell,p+q}, \nu_j \in V^{p+q},$$

$$\|g\|_{\Gamma_c^{\ell,p+q}(\alpha,G)} \leq 1, \|\nu_j\|_{C^{p+q}(U_{\alpha,G})} \leq 1\}.$$

where, for $\nu \in V^s(\alpha, G)$, $\|\nu\|_{C^s(U_{\alpha,G})} = \sup_{\alpha,i} \|\langle \nu, e_{\alpha,i} \rangle \circ \Theta_\alpha^{-1}\|_{C^s(\Theta_\alpha(U_{\alpha,G})}$.
For all $p \in \mathbb{N}$, $q \in \mathbb{R}_+$, $\ell \in \{0, \cdots, 2\}$, we define the spaces $\mathcal{B}^{p,q,\ell}$ as the closure of the $\mathcal{C}^\infty$ ℓ forms with respect to the norm

$$\|h\|_{p,q,\ell} = \sup_{n \leq p} \sup_{J \in \mathbb{U}_{\rho,L,n,q,\ell}} J(h).$$

[4] We use the notation $B_d(x, r) = \{y \in \mathbb{R}^d \ : \ \|y - x\| < r\}$.

The main result of this section consists in the following Theorem which provides a rather precise characterisation of the spectrum of the action on one forms, which is well known to be related to the measure of maximal entropy and thus plays the same role of the operators $\mathcal{L}_0$ in the previous sections.

Theorem 6.11. *For each $\varepsilon > 0$, for p, q large enough,*

$$\sigma_{\mathcal{B}^{p,q,1}}(f_*) \cap \{z \in \mathbb{C} \;:\; |z| > \kappa\} = \{e^{h_{top}}\}$$
$$\{e^{-h_{top}}, e^{h_{top}}\} \cup (\sigma_{\mathcal{B}^{p+1,q-1,0}}(f_*) \setminus \{z \in \mathbb{C} \;:\; |z| < \varepsilon\}) \subset \sigma_{\mathcal{B}^{p,q,1}}(f_*)$$
$$\sigma_{\mathcal{B}^{p,q,1}}(f_*) \subset \{z \in \mathbb{C} \;:\; |z| < \varepsilon\} \cup \{e^{-h_{top}}, e^{h_{top}}\} \cup \sigma_{\mathcal{B}^{p+1,q-1,0}}(f_*) \cup \sigma_{\mathcal{B}^{p-1,q+1,0}}(\mathcal{L}),$$

where $\kappa \in (0,1)$ is the maximum of the modulus of the second eigenvalue of f_* and $\mathcal{L}$ acting on $\mathcal{B}^{p+1,q-1,0}$ and $\mathcal{B}^{p-1,q+1,0}$, respectively.

Remark 6.12. In fact, we conjecture that, for $\lambda^{\min\{p,q\}} > \varepsilon^{-1}$,

$$\sigma_{\mathcal{B}^{p,q,1}}(f_*) \setminus \{z \in \mathbb{C} \;:\; |z| < \varepsilon\} = \left[\{e^{-h_{\text{top}}}, e^{h_{\text{top}}}\} \cup \sigma_{\mathcal{B}^{p+1,q-1,0}}(f_*) \cup \sigma_{\mathcal{B}^{p-1,q+1,0}}(\mathcal{L})\right] \setminus \{1\},$$

see Remark 6.22. This would be consistent with the fact that, by duality, the spectrum of $\mathcal{L}$ equals the spectrum of f_*^{-1} and that the spectra of f_* on forms determine the Ruelle zeta function, see [73], and the latter is described in term of periodic orbits, which are the same for f and f^{-1}. Accordingly, one expects a symmetry between the spectra of f_* and f_*^{-1}.[5]

The next section is devoted to the proof of the above Theorem, while in Section 6.0.7 we present a minimalistic discussion of cohomology in the spaces $\mathcal{B}^{p,q,1}$ and Section 7 is devoted to comments on the implications of such a Theorem and a comparison with existing results.

6.0.5 Proof of Theorem 6.11

We start with some preliminary results establishing minimal information about Hodge duality and exterior differentials in our spaces of currents $\mathcal{B}^{p,q,\ell}$.

Lemma 6.13. *The Hodge duality map $\Phi h := \star h = h\omega_0$, between zero forms and two forms, extends to a bounded isomorphism between $\mathcal{B}^{p,q,0}$ and $\mathcal{B}^{p,q,2}$ and $\Phi\mathcal{L} = f_*\Phi$. In particular, $\sigma_{\mathcal{B}^{p,q,2}}(f_*) = \sigma_{\mathcal{B}^{p,q,0}}(\mathcal{L})$.*

Proof. By equation (6.8), for each smooth zero form h, $\Phi h = h\omega_0$. Thus equation (6.4) implies $\Phi\mathcal{L}h = f_*\Phi h$ for each smooth zero form. The injectivity follows since $\mathcal{B}^{p,q,0}, \mathcal{B}^{p,q,2}$ are isomorphic to a subspace of the space of currents, see [46, Lemma 3.10], and the extension of Φ to the current is an isomorphism. The result then follows by proving that Φ is a bounded operator. For each multi-index α, $|\alpha| = p$, smooth two form ω and zero form h we have

$$\left|\int_W \langle \omega, \partial^\alpha \Phi(h)\rangle\right| \leq \sum_{\beta+\gamma=\alpha} \left|\int_W \langle \omega, \partial^\beta \omega_0\rangle \partial^\gamma h\right| \leq C_\sharp \|\omega\|_{\mathcal{C}^{q+p}(W)} \|h\|_{\mathcal{B}^{p,q,0}}$$

from which the claim follows. □

[5] Note that if f_* acts on some Banach space, then here one considers f_*^{-1} acting on its dual, so the relation is not obvious a priori. Indeed, one does not necessarily expect f_*^{-1} to be a bounded operator when acting on a Banach space on which f_* is bounded.

Lemma 6.14. *The exterior derivative d extends to a bounded operator $\mathcal{B}^{p,q,\ell} \to \mathcal{B}^{p-1,q+1,\ell+1}$.*[6] *If $h \in \mathcal{B}^{p,q,0}$, $\psi \in \mathcal{C}^\infty(M, \mathbb{R}_+)$, with the interior of $\operatorname{supp}(\psi)$ connected, and $\psi dh = 0$, then there exists $c \in \mathbb{C}$ such that $\psi(h - c) = 0$.*[7] *Finally, $d(\mathcal{B}^{p,q,0})$ is closed in $\mathcal{B}^{p-1,q+1,1}$.*

Proof. If h is an ℓ form, then, for each $\ell + 1$ form ω and multi-index α, $|\alpha| = p - 1$, we have that there exists a constant $C_\ell > 0$ such that

$$\left| \int_W \langle \omega, \partial^\alpha dh \rangle \right| \leq C_\ell \|\omega\|_{\mathcal{C}^{p+q}(W)} \|h\|_{p,q,\ell}$$

from which it follows $\|dh\|_{p-1,q+1,\ell+1} \leq C_\ell \|h\|_{p,q,\ell}$.

Next, let h, ψ be such that $\psi dh = 0$. Note that $\mathcal{B}^{p,q,0}$ is isomorphic to a subspace of the space of distributions $(\mathcal{C}^{p+q})'$, see [46, Lemma 3.10]. Let $K = \operatorname{supp}\psi$ and $U = \mathring{K}$, note that U is connected by hypothesis. Thus for each smooth local function φ, $\operatorname{supp}\varphi \subset U$, and disintegration of ω along a smooth foliation $\{W_t\} \subset \Sigma$, we have[8]

$$\int_M \varphi \partial_{x_i} h = \int dt \int_{W_t} \langle \frac{\varphi}{\psi} dx_i, \psi dh \rangle = 0. \tag{6.10}$$

It follows that $\partial_{x_i} h = 0$ as a distribution on U, hence $h = c$ on U, for some $c \in \mathbb{C}$. That is $\psi(h - c) = 0$ on M. From [46, Lemma 3.10], again, it follows that $\psi(h - c) = 0$ as an element of $\mathcal{B}^{p,q,0}$.

To conclude the Lemma we want to prove that $d(\mathcal{B}^{p,q,0})$ is closed in $\mathcal{B}^{p-1,q+1,1}$. Let us suppose that $\omega_n \to \omega$, in $\mathcal{B}^{p-1,q+1,1}$, with $\omega_n \in d(\mathcal{B}^{p,q,0})$. That is, there exists $\Xi_n \in \mathcal{B}^{p,q,0}$ such that $\omega_n = d\Xi_n$. Let $\widehat{\Xi}_n = \Xi_n - \int_M \Xi_n$. Then, for each function φ supported in a chart $(U_\alpha, \Theta_\alpha)$ we can write

$$\int \varphi \widehat{\Xi}_n = \int_M \left(\varphi - \int_M \varphi \right) \Xi_n.$$

Let $\bar{x}$ be such that $\int_M \varphi = \varphi(\bar{x})$, then[9]

$$\begin{aligned} \int \varphi \widehat{\Xi}_n &= \int_{U_\alpha} dx \int_0^1 dt \frac{d}{dt} \varphi(\bar{x} + (x - \bar{x})t) \Xi_n(x) \\ &= -\sum_{i=1}^{2} \int_M \varphi(\bar{x} + (x - \bar{x})t) \langle (x_i - \bar{x}_i), \partial_{x_i} \Xi_n(x) \rangle. \end{aligned}$$

Hence, setting $\Psi_t(x) = -\sum_{i=1}^2 \varphi(\bar{x} + (x - \bar{x})t)(x_i - \bar{x}_i) dx_i$, we have, recalling equation 6.6,

$$\int \varphi \widehat{\Xi}_n = \int_0^1 dt \int_M \langle \Psi_t, \omega_n \rangle.$$

Arguing similarly for $\partial^\alpha \widehat{\Xi}_n$ it follows that $\widehat{\Xi}_n$ is a Cauchy sequence. Let Ξ be the limit, then, by the continuity of d, $d\Xi = \omega$, hence $\omega \in d(\mathcal{B}^{p,q,0})$. □

[6] With a slight abuse of notation we will call such an extension d as well.

[7] This essentially says that closed anisotropic zero currents are constant. Since for zero currents being closed and being harmonic is the same, this is a little piece of Hodge theory, all that is presently needed. Yet, it would be clearly useful to develop the Hodge theory in the context of anisotropic spaces.

[8] Since the foliation is smooth the Jacobian J of the disintegration is a smooth function and $\hat{\varphi} = J\varphi$. Note that $\frac{\hat{\varphi}}{\psi}$ is a smooth function on W_t.

[9] To simplify notation we do not write explicitly the change of coordinates Ξ_α.

Spectral radius and essential spectral radius of f_* and $\mathcal{L}$

The first step in the study of the operators $f_*, \mathcal{L}$ is the following.

Lemma 6.15. *The action of f_* on ℓ form extends to a linear bounded operator from $\mathcal{B}^{p,q,\ell}$ to itself. With a slight abuse of notation we use f_* for the action on each $\mathcal{B}^{p,q,\ell}$. Then, we have*

$$
\begin{aligned}
&\|f_*^n h\|_{\mathcal{B}^{0,q,0}} \leq C_\sharp \|h\|_{\mathcal{B}^{0,q,0}}\\
&\|f_*^n h\|_{\mathcal{B}^{0,q,1}} \leq C_\sharp e^{h_{top} n} \|h\|_{\mathcal{B}^{0,q,1}}\\
&\|f_*^n h\|_{\mathcal{B}^{p,q,0}} \leq C_\sharp \lambda^{-np} \|h\|_{\mathcal{B}^{p,q,0}} + C_\sharp \|h\|_{\mathcal{B}^{p-1,q+1,0}}\\
&\|f_*^n h\|_{\mathcal{B}^{p,q,1}} \leq C_\sharp e^{h_{top} n} \lambda^{-np} \|h\|_{\mathcal{B}^{p,q,1}} + C_\sharp e^{h_{top} n} \|h\|_{\mathcal{B}^{p-1,q+1,1}}\\
&\|\mathcal{L}^n h\|_{\mathcal{B}^{p,q,0}} \leq C_\sharp \lambda^{-np} \|h\|_{\mathcal{B}^{p,q,0}} + C_\sharp \|h\|_{\mathcal{B}^{p-1,q+1,0}}.
\end{aligned}
$$

Proof. To start with let $h \in \mathcal{C}^\infty(M, \mathbb{C})$ be a function, then

$$
\int_W \langle \varphi, f_*^n h \rangle = \int_W \langle \varphi, h \circ f^{-n} \rangle = \int_{f^{-n} W} \langle \varphi \circ f^n, h \lambda_n^s \rangle
$$

where $\lambda_n^s(x)$ is the contraction of f^n in the direction $T_* W$ at the point x. We can divide $f^{-n}W$ in a collection $\{W_i\} \subset \Sigma$. Let $\{\vartheta_i\}$ be a smooth partition of unity subordinated to $\{W_i\}$. If $\lambda_{n,i}^s = \min_{x \in W_i} \lambda_n^s(x)$, then the usual distortion arguments implies, that for all $x \in W_i$, $C_\sharp \lambda_{n,i}^s \leq \lambda_n^s(s) \leq C_\sharp \lambda_{n,i}^s$, thus, integrating $|f^n(W_i)| = \int_{W_i} \lambda_{n,i}^s \geq C_\sharp \lambda_{n,i}^s \delta$. In addition, for all $q \in \mathbb{N}$, $\|\lambda_n^s(s)\|_{\mathcal{C}^q(W_i)} \leq C_\sharp \lambda_{n,i}^s$. Accordingly,

$$
\begin{aligned}
\left| \int_W \langle \varphi, f_*^n h \rangle \right| &\leq \sum_i \left| \int_{W_i} \langle \vartheta_i \lambda_n^s \varphi \circ f^n, h \rangle \right| \\
&\leq C_\sharp \sum_i |f^n(W_i)| \delta^{-1} \|\varphi \circ f^n\|_{\mathcal{C}^q(W_i)} \|h\|_{0,q,0} \\
&\leq C_\sharp \|\varphi\|_{\mathcal{C}^q(W)} \|h\|_{0,q,0}
\end{aligned}
$$

which, by density, proves the first inequality of the Lemma. Next, recalling equation 6.6, we have, for each $v = \pi(\omega), h$ a $\mathcal{C}^\infty$ one form,

$$
\left| \int_W \langle \omega, f_*^n h \rangle \right| \leq \left| \int_W f_*^n h(v) \right| = \left| \int_W h_{f^{-n}(x)}(d_x f^{-n} v(x)) dx \right|.
$$

Setting $v_n(x) = d_{f^n(x)} f^{-n} v(f^n(x))$, by the usual distortion arguments we have $\|v_n\|_{\mathcal{C}^q(W_i)} \leq C_\sharp (\lambda_{n,i}^s)^{-1} \|v\|_{\mathcal{C}^q(W)}$, hence

$$
\begin{aligned}
\left| \int_W \langle \omega, f_*^n h \rangle \right| &\leq C_\sharp \sum_i \left| \int_{W_i} h(\vartheta_i \lambda_n^s v_n) \right| \leq C_\sharp \sum_i |W_i| \delta^{-1} \|h\|_{0,q,1} \\
&\leq C_\sharp e^{h_{top} n} \|\omega\|_{\mathcal{C}^q} \|h\|_{0,q,1}
\end{aligned}
$$

where, in the last line, we have used $|f^{-n}W| \sim e^{h_{\text{top}} n}$, see [46, Appendix D]. Taking the sup on W and ω the second inequality of the Lemma follows.

The next two inequalities are proven, similarly, as done in [46, Lemma 4.7], while the last follows by [44, Lemma 2.2] taking into account Lemma 6.13. □

We are now able to obtain a first information on the peripheral spectrum.

Lemma 6.16. *For p, q large enough, the spectra of f_* on $\mathcal{B}^{p,q,0}$ and on $\mathcal{B}^{p,q,2}$ are contained in $\{1\} \cup \{z \in \mathbb{C} : |z| < \kappa\}$ for some $\kappa < 1$. The eigenvectors associated to the eigenvalue 1 are the constant function 1 and the measure μ_{SRB} respectively.*

Proof. By Lemma 6.13 the action of f_* on $\mathcal{B}^{p,q,2}$ is conjugated to the action of $\mathcal{L}$ on $\mathcal{B}^{p,q,0}$, thus they have the same spectrum. But [44] implies that there exists $\kappa \in (0,1)$ such that $\sigma_{\mathcal{B}^{p,q,0}}(\mathcal{L}) \subset \{1\} \cup \{z \in \mathbb{C} : |z| \leq \kappa\}$ and one is a simple eigenvalue. This proves the Lemma for $\mathcal{B}^{p,q,2}$.

Let us discuss $\mathcal{B}^{p,q,0}$. Lemma 6.15 and Hennion's Theorem 2.23 imply that the radius of the essential spectrum of f_* acting on $\mathcal{B}^{p,q,0}$ is at most λ^{-1} while the spectral radius is one, moreover the operator is power bounded. Accordingly, if f_* has no eigenvalue on the unit circle apart from 1 and 1 is a simple eigenvalue, then there exists a κ that satisfies the Lemma. Thus, we need only study eigenvalues of the form $e^{i\theta}$. Since the operator is power bounded, to such a maximal eigenvalue it cannot be associated a Jordan block, hence their geometric and algebraic multiplicity coincide. Hence, we have the spectral decomposition

$$f_* = \sum_j e^{i\theta_j} \Pi_j + Q$$

where $\theta_j \in \mathbb{R}$, $\Pi_i \Pi_j = \delta_{ij} \Pi_j$, $\Pi_j Q = Q \Pi_j = 0$ and $\|Q^n\| \leq C_\sharp \kappa^n$. Suppose that $e^{i\theta} \in \sigma_{\mathcal{B}^{p,q,0}}(f_*)$, $\theta \in \mathbb{R} \setminus \{0\}$, then there exists $h \in \mathcal{B}^{p,q,0} \setminus \{0\}$ such that $f_* h = e^{i\theta} h$, and, by the spectral decomposition, there exists $h_0 \in \mathcal{C}^\infty$ such that

$$h = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-i\theta k} f_*^k h_0.$$

It follows that, for all $\varphi \in \mathcal{C}^\infty(M, \mathbb{C})$,[10]

$$\int_M \varphi h \omega_0 = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-i\theta k} \int_M \varphi f_*^k h_0 \omega_0 = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-i\theta k} \int_M \varphi \cdot h_0 \circ f^{-k} \omega_0.$$

But f^{-1} is also a transitive Anosov diffeomorphism with its SRB measure, call it μ^-_{SRB}, then

$$\lim_{k\to\infty} \int_M \varphi \cdot h_0 \circ f^{-k} \omega_0 = \int_M \varphi \omega_0 \int_M h \mu^-_{\mathrm{SRB}}$$

which implies

$$\int_M \varphi h \omega_0 = 0$$

and since $\mathcal{B}^{p,q,0}$ is a space of distributions, see [46], it follows $h = 0$ contrary to the hypothesis. We are left with case $\theta = 0$, that is the eigenvalue 1. Since $1 \circ f^{-1} = 1$, one is an eigenvalue, we want to prove that it is simple. Let $f_* h = h$, and let h_0 as before,

$$\left| \int_M \varphi h \omega_0 \right| \leq \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_M |\varphi| \cdot |h_0| \circ f^{-k} \omega_0 \leq \|\varphi\|_{L^1(\omega_0)} \|h_0\|_{\mathcal{C}^0}.$$

Thus $h \in L^\infty(\omega_0)$ (the dual of $L^1(\omega_0)$), and $h = h \circ f^{-1}$, ω_0-almost surely. On the other hand, since ω_0 is ergodic for the Anosov map, it follows that h is almost surely constant. Thus 1 is simple point spectrum for f_* acting on $\mathcal{B}^{p,q,0}$. □

[10]Note that the integral on M can be decomposed as an integral over elements of Σ, which are continuous functional in the $\mathcal{B}^{p,q}$ norms, hence we can exchange the limit with the integral.

The peripheral spectrum of f_* acting on $\mathcal{B}^{p,q,1}$

Here, we start a more in depth study of $\sigma_{\mathcal{B}^{p,q,1}}(f_*)$.

Lemma 6.17. *The spectrum of f_* on $\mathcal{B}^{p,q,1}$ contains $e^{h_{top}}$, which is also the spectral radius. In addition, the essential spectral radius is bounded by $e^{h_{top}}\lambda^{-\min\{p,q\}}$. The eigenvector associated to $e^{h_{top}}$ is the Bowen-Margulis measure and, together with the dual eigenvector defines the measure of maximal entropy.*

Proof. The statement on the spectral radius and essential spectrum follows from Lemma 6.15 and Hennion's Theorem 2.23. Next, if ν is an eigenvalue of f_*, $|\nu| = e^{h_{top}}$, then, by Lemma 6.15, $\nu^{-n}f_*$ is power bounded, hence it cannot be associated to a Jordan block. Let $h \in \mathcal{B}^{p,q,1}$ be an eigenvalue, then, by Lemma 6.15 again, $\|h\|_{\mathcal{B}^{p,q,1}} \leq C_\sharp \|h\|_{\mathcal{B}^{0,p+q,1}}$.

Next, let $\bar{v}^u(x) \in \mathcal{C}(x)$ be a smooth normalised vector field. Then, for each $\omega \in \mathcal{C}^{p+q}$, $W \in \Sigma$ and $n \in \mathbb{N}$ let $v = \pi(\omega)$ and $v = w + v^u$ where v belongs to the tangent space of W and $v^u(x) = \alpha(f^{-n}(x))df^n_{f^{-n}(x)}\bar{v}^u(f^{-n}(x))$. Note that $\|\alpha\|_{\mathcal{C}^{p+q}(f^{-n}(W))} \leq \lambda^{-n}C_\sharp\|\omega\|_{\mathcal{C}^{p+q}}$. Thus,

$$\begin{aligned}\left|\int_W \langle \omega, h\rangle\right| &= \left|\int_W \langle \omega, \nu^{-n}f_*^n h\rangle\right| = \left|\nu^{-n}\int_W f_*^n h(v)\right| \\ &\leq \left|\int_W \nu^{-n}h_{f^{-n}(x)}(d_x f^{-n}w(x))dx\right| + \left|\int_W \nu^{-n}\alpha(f^{-n}(x))h_{f^{-n}(x)}(\bar{v}^u(f^{-n}(x))dx\right| \\ &\leq C_\sharp \sum_i \int_{W_i} \vartheta_i(x)\nu^{-n}h_x(\bar{w}_i(x))dx + C_\sharp\lambda^{-n}\|h\|_{\mathcal{B}^{0,p+q,1}}\|v\|_{\mathcal{C}^{p+q}}\end{aligned}$$

where $\bar{w}_i$ belongs to the tangent space of W_i and $\|\bar{w}_i\|_{\mathcal{C}^{p+q}} \leq \|w\|_{\mathcal{C}^0} + \lambda^{-n}\|w\|_{\mathcal{C}^{p+q}}$. Thus,

$$\begin{aligned}\left|\int_W \langle \omega, h\rangle\right| &\leq C_\sharp|f^{-n}W|\delta^{-1}e^{-h_{top}n}\|h\|_{\mathcal{B}^{0,p+q,1}}\|w\|_{\mathcal{C}^0} + C_\sharp\lambda^{-n}\|h\|_{\mathcal{B}^{0,p+q,1}}\|v\|_{\mathcal{C}^{p+q}} \\ &\leq C_\sharp\delta^{-1}\|h\|_{\mathcal{B}^{0,p+q,1}}\|\omega\|_{\mathcal{C}^0} + C_\sharp\lambda^{-n}\|h\|_{\mathcal{B}^{0,p+q,1}}\|\omega\|_{\mathcal{C}^{p+q}}\end{aligned}$$

where, in the last line, we have used the estimate on the growth of invariant manifolds, see [46, Appendix C] for details. Taking the limit $n \to \infty$ and the sup in W and ω yields

$$\|h\|_{\mathcal{B}^{0,0,1}} \leq C_\sharp\|h\|_{\mathcal{B}^{p,q,1}}.$$

Next, let v^s be the normalised stable direction. Then, setting $\bar{\phi}(x) = \ln|d_{f(x)}f^{-1}v^s(f(x))|$,

$$d_x f^{-n}v^s(x) = e^{\sum_{k=0}^{n-1}\bar{\phi}\circ f^{-n+k}(x)}v^s(f^{-n}(x)).$$

We can then define the transfer operator

$$\mathcal{L}_* g(x) = g \circ f^{-n}(x)e^{\bar{\phi}(x)}. \tag{6.11}$$

Defining the map $\Gamma : \mathcal{B}^{0,0,1} \to \mathcal{M}$, the space of signed measures, by $\Gamma(h) = h(v^s)\omega_0$, we have

$$\int_M f_*(h)(gv^s) = \int_M g\mathcal{L}_*\Gamma(h).$$

In [45] is proven that $\mathcal{L}_*$ is has maximal eigenvalue $e^{-h_{top}}$ and the associated eigenvector is the Bowen-Margulis measure. This concludes the Lemma. □

Deeper in the spectrum of f_* acting on $\mathcal{B}^{p,q,1}$

By Lemma 6.14 we can extend the de Rham cohomology to the currents in the spaces $\mathcal{B}^{p,q,\ell}$. In other words we can call *closed* the elements $\omega \in \mathcal{B}^{p,q,\ell}$ such that $d\omega = 0$ and *exact* the ones for which it exists $\alpha \in \mathcal{B}^{p+1,q-1,\ell-1}$ such that $\omega = d\alpha$.

Remark 6.18. By equation (6.5) and Lemma 6.14 it follows that f_* sends closed currents into closed currents and exact currents into exact currents. Hence f_* induces an action in cohomology (of the $\mathcal{B}^{p,q,\ell}$ currents), let us call it $f_\sharp$.

The next result shows that such a cohomology (let us call it *anisotropic cohomology*) is relevant to our problem.

Lemma 6.19. *If $\nu \in \sigma_{\mathcal{B}^{p,q,1}}(f_*)$, and $\omega \in \mathcal{B}^{p,q,1}$ are such that $f_*\omega = \nu\omega$ and $|\nu| > e^{h_{top}}\lambda^{-\min\{p,q\}}$, then either ω is not exact or $\nu \in \sigma_{\mathcal{B}^{p+1,q-1,0}}(f_*)\setminus\{1\}$. Moreover, $\sigma_{\mathcal{B}^{p+1,q-1,0}}(f_*)\setminus \{1\} \subset \sigma_{\mathcal{B}^{p,q,1}}(f_*)$. If $\nu \in (\sigma_{\mathcal{B}^{p,q,1}}(f_*) \setminus \sigma_{\mathcal{B}^{p-1,q+1,0}}(\mathcal{L})) \cup \{1\}$, then for each $\omega \in \mathcal{B}^{p,q,1}$ such that $f_*\omega = \nu\omega$ we have $d\omega = 0$.*

Proof. To start with note that, by Lemma 6.17, ν must belong to the point spectrum.

Let $\nu \in (\sigma_{\mathcal{B}^{p,q,1}}(f_*) \setminus \sigma_{\mathcal{B}^{p,q,0}}(f_*)) \cup \{1\}$ and $\omega \in \mathcal{B}^{p,q,1}$ such that $f_*\omega = \nu\omega$ and suppose that ω is exact. Thus, there exists $h \in \mathcal{B}^{q-1,p+1,0}$ such that $dh = \omega$. This implies

$$\nu dh = f_*dh = df_*h.$$

That is $d(f_*h - \nu h) = 0$. It follows by Lemma 6.14 that $f_*h = \nu h + c$. By a change of variable it follows that the dual $(f_*)'$ of f_* is given by the transfer operator $\mathcal{L}_{f^{-1}}$ associated to the map f^{-1}. Since f^{-1} is Anosov as well Lemmata 6.15 and 6.16 apply and the measure μ^-_{SRB} associated to f^{-1} belongs to the dual of $\mathcal{B}^{p,q,0}$. Since $f_*1 = 1$ and the space $\mathbb{V} = \{h : \int_M h d\mu^-_{\text{SRB}} = 0\}$ is invariant for f_*, it is natural to write $h = \alpha + g$ with $\alpha \in \mathbb{C}$ and $g \in \mathbb{V}$. Then, we have

$$c + \nu\alpha + \nu g = \alpha + f_*g.$$

Applying μ^-_{SRB} to the above implies $c = \alpha(1-\nu)$, hence $\nu g = f_*g$. The only possibility is then $\nu = 1$ but the associated eigenvector would be $1 \notin \mathbb{V}$, it follows $g = 0$. But then $\omega = dh = d\alpha = 0$. Hence, ω cannot be exact. The inclusion of the spectra is obvious.

If $f_*\omega = \nu\omega$ and $d\omega = h\omega_0$, by Lemma 6.13 we have $\mathcal{L}h = \nu h$. Accordingly, either $\nu \in \sigma_{\mathcal{B}^{p-1,q+1,0}}(\mathcal{L})$ or $h = 0$, that is $d\omega = 0$. On the other hand, if $\nu = 1$, then $h\omega_0 = \mu_{\text{SRB}}$. Hence, $d\omega = \mu_{\text{SRB}}$ and

$$\int_M \mu_{\text{SRB}} = \int_M d\omega = 0$$

which is impossible since μ_{SRB} is a positive measure. Accordingly, it must be $d\omega = 0$, that is, again, the form is closed. □

To conclude we need a theory of anisotropic de Rham cohomology, such a general theory goes a beyond our present scopes so we will develop only the minimal version needed here. This is contained in Section 6.0.7, and in particular in Lemma 6.25 which states that the anisotropic cohomology of one forms is isomorphic to standard de Rham cohomology. In particular, this implies that the vector space of the equivalence classes is finite dimensional, hence $f_\sharp$, defined in Remark 6.18, has only point spectrum, let us call Ω the spectrum of $f_\sharp$ when acting on one forms.

Next, we want to identify Ω. As stated in Remark 6.9 this is the only place where we use that our map is topologically conjugated to the linear model.

Lemma 6.20. *We have* $\Omega = \{e^{-h_{top}}, e^{h_{top}}\}$.

Proof. Lemma 6.25 implies that the anisotropic de Rham cohomology for one forms is a topological invariant, hence so is $f_\sharp$. Since our map is conjugated to a linear model $f_\sharp$ is conjugated to the action of the linear model on homology. The Lemma follows by a direct computation, see [52, Section 3.2-e] for details. □

The following Lemma concludes the proof of Theorem 6.11.

Lemma 6.21. *For each* $\varepsilon > 0$*, if* p, q *are large enough, we have*

$$\Big[\Omega \cup \sigma_{\mathcal{B}^{p+1,q-1,0}}(f_*) \setminus \Big(\{1\} \cup \{z \in \mathbb{C} \ : \ |z| < \varepsilon\}\Big)\Big] \subset \sigma_{\mathcal{B}^{p,q,1}}(f_*)$$
$$\sigma_{\mathcal{B}^{p,q,1}}(f_*) \subset \Big[\{z \in \mathbb{C} \ : \ |z| < \varepsilon\} \cup \Omega \cup \sigma_{\mathcal{B}^{p+1,q-1,0}}(f_*) \cup \sigma_{\mathcal{B}^{p-1,q+1,0}}(\mathcal{L})\Big] \setminus \{1\}.$$

Proof. Lemma 6.17 implies that if p, q are large enough we have to worry only about point spectrum.

Thus, if $\nu \in \sigma_{\mathcal{B}^{p+1,q-1,0}}(f_*)$ then there exists $\theta \in \mathcal{B}^{p+1,q-1,0}$ such that $f_*\theta = \nu\theta$. This implies that $f_* d\theta = \nu d\theta$ so either $d\theta = 0$, but then by Lemma 6.14 we have h constant and $\nu = 1$, or $\nu \in \sigma_{\mathcal{B}^{p,q,1}}(f_*)$.

If $\nu \in \Omega$, the spectrum of $f_\sharp$ (defined in Remark 6.18), then it means that there exists $\omega \in \mathcal{B}_0^{p,q,1}$ and $\psi \in \mathcal{B}^{p+1,q-1,0}$ such that $f_*\omega = \nu\omega + d\psi$, that is $f_\sharp[\omega] = \nu[\omega]$, where $[\omega] \neq 0$ is the equivalence class of ω. If $\nu \notin \sigma_{\mathcal{B}^{p+1,q-1,0}}(f_*)$, we can define $\theta = (\nu - f_*)^{-1}\psi$ and

$$(\nu - f_*) d\theta = d\psi.$$

But then $f_*(\omega + d\theta) = \nu(\omega + d\theta)$ which implies $\nu \in \sigma_{\mathcal{B}_0^{p,q,1}}(f_*)$ unless $\omega + d\theta = 0$. But the latter possibility would imply that ω is exact, that is $[\omega] = 0$, contrary to the assumption. This proves the first inclusion of the Lemma.

To prove the second inclusion note that if $f_*\omega = \nu\omega$, $\omega \in \mathcal{B}^{p,q,1}$ and $\nu \notin \sigma_{\mathcal{B}^{p-1,q+1,0}}(\mathcal{L})\setminus\{1\}$, then the last part of the Lemma 6.19 implies $d\omega = 0$. Then $f_\sharp[\omega] = \nu[\omega]$, thus either $\nu \in \Omega$ or $[\omega] = 0$, i.e. ω is exact. But if $\nu \notin \sigma_{\mathcal{B}^{p+1,q-1,0}}(f_*) \setminus \{1\}$ the first part of Lemma 6.19 implies that ω is not exact, hence $[\omega] \neq 0$. Since Lemma 6.20 implies $1 \notin \Omega$, the Lemma follows. □

Remark 6.22. It is conceivable that Lemma 6.21 could be upgraded to an equality. Indeed, suppose that for a two current $\int_M \omega = 0$ implies that there exist a one current θ such that $\omega = d\theta$.[11] Then if $f_*\omega = \nu\omega$, $\nu \neq 1$ and $\omega \not\equiv 0$, we have $\int_M \omega = 0$ thus we can write $\omega = d\theta$ and $d(\nu\theta - f_*\theta) = 0$. Thus $\nu\theta - f_*\theta = \psi$ with $d\psi = 0$. Hence, if $\nu \notin \sigma_{\mathcal{B}^{p,q,1}}(f_*)$, we have $\theta = (\nu - f_*)^{-1}\psi$. Since $d(z - f_*)^{-1}\psi$ is a meromorphic function and for large z the Von Newman expansion implies the it is zero, we have $d\theta = 0$, a contradiction. Hence the second inclusion of the Lemma 6.21 would be an equality.

6.0.6 Application to the measure of maximal entropy

In this section we prove Theorem 6.10.

Lemma 6.17 implies that there exists $\ell_\star \in (\mathcal{B}^{p,q,1})'$ and $h_\star \in \mathcal{B}^{p,q,1}$ such that $f_* h_\star = e^{h_{\rm top}} h_\star$ and $\ell_\star(f_*\omega) = e^{h_{\rm top}}\ell_\star(\omega)$, for all $\omega \in \mathcal{B}^{p,q,1}$. In addition, $\ell_\star(\varphi h_\star) = \mu_{\rm BM}(\varphi)$. Lemmata 6.21 and 6.16 imply that the rest of the spectrum is contained in $\{z \in \mathbb{C} \ : \ |z| < \kappa\}$ for some $\kappa \in (0, 1)$. It follows that the spectral decomposition $f_* = e^{h_{\rm top}} h_\star \otimes \ell_\star + \mathcal{Q}$ with $\ell_\star \mathcal{Q} = 0$,

[11] This is equivalent to studying the cohomology for two forms.

$\mathcal{Q}h_\star = 0$, $\ell(h) = 1$ and $\|\mathcal{Q}^n\|_{p,q,1} \le C_\sharp \kappa^n$. Also note that the multiplication by a smooth function is a bounded operator. Thus

$$\begin{aligned}\int_M g\circ f^n h d\mu_{\text{BM}} &= \ell_\star(g\circ f^n h h_\star) = e^{-nh_{\text{top}}}\ell_\star(f_*^n(g\circ f^n h h_\star)) = e^{-nh_{\text{top}}}\ell_\star(g f_*^n(h h_\star))\\ &= \ell_\star(g h_\star)\ell_*(h h_\star) + e^{-nh_{\text{top}}}\ell_\star(g\mathcal{Q}^n(h h_\star)).\end{aligned}$$

It follows that, for r large enough,

$$\left|\int_M g\circ f^n h d\mu_{\text{BM}} - \int_M g d\mu_{\text{BM}}\int_M h d\mu_{\text{BM}}\right| \le C_\sharp \|g\|_{\mathcal{C}^r}\|h\|_{\mathcal{C}^r} e^{-nh_{\text{top}}}\kappa^n.$$

6.0.7 Anisotropic de Rham cohomology

While to develop a theory of anisotropic de Rham cohomology as well as the relative Hodge theory may certainly be of interest, in this section we will develop only the bare minimum necessary to our needs and we will keep the arguments as elementary as possible.

Without loss of generality we can, and will, assume that there exist *good covers* $\{U_\alpha^+\}$, and $\{U_\alpha\}$ such that $U_\alpha^+ \supset \overline{U}_\alpha$, and a partition of unity $\{\psi_\alpha\}$ subordinated to $\{U_\alpha\}$. Also let $\{\psi_\alpha^+\}$ be such that $\text{supp}(\psi_\alpha^+) \subset U_\alpha^+$ and $\psi_\alpha^+|_{U_\alpha} = 1$.[12]

Lemma 6.23. *If $h \in \mathcal{B}^{p,q,1}$ is closed then, for each α, there exists $H_\alpha, H_\alpha^+ \in \mathcal{B}^{p+1,q-1,0}$ such that $dH_\alpha = h\psi_\alpha + H_\alpha^+ d\psi_\alpha$ and $H_\alpha = H_\alpha^+\psi_\alpha$.*

Proof. For each U_α let us choose $x_\alpha \in \Theta_\alpha(U_\alpha^+ \setminus \text{supp}\psi_\alpha^+)$. We start assuming that h is a smooth one form and we define, for all $x \in U_\alpha$,

$$\bar{H}_\alpha(x) = \int_0^1 \Theta_{\alpha *} h_{x_\alpha(1-t)+tx}(x - x_\alpha)dt. \tag{6.12}$$

Also, for simplicity of notation, we confuse h and $\Theta_{\alpha *}h =: \sum_i h_i dx_i$ and set $\gamma_{\text{lin},x}(t) = x_\alpha(1-t) + tx$. Then

$$\begin{aligned}\partial_{x_i}\bar{H}_\alpha(x) &= \sum_{k=1}^2 \int_0^1 [t(\partial_{x_i}h_k)\circ\gamma_{\text{lin},x}(t)(x-x_\alpha)_k + h_i\circ\gamma_{\text{lin},x}(t)]\\ &= \sum_{k=1}^2 \int_0^1 t(\partial_{x_k}h_i)\circ\gamma_{\text{lin},x}(t)(x-x_\alpha)_k + h_i\circ\gamma_{\text{lin},x}(t)\\ &\quad - \int_0^1 t\, dh(x-x_\alpha, e_i)\\ &= \int_0^1 \frac{d}{dt}[th_i\circ\gamma_{\text{lin},x}(t)] - \int_0^1 t\, d_{\gamma_{\text{lin},x}(t)}h(x-x_\alpha,e_i)\\ &= h_i(x) - \int_0^1 t\, d_{\gamma_{\text{lin},x}(t)}h(x-x_\alpha,e_i).\end{aligned} \tag{6.13}$$

Thus, if h is a closed form, then we have $d\bar{H}_\alpha = h$.

[12] Recall that a good cover is a cover such that, for each collection $\mathcal{A}$ of indexes, $\cap_{\alpha\in\mathcal{A}}U_\alpha$ is contractible.

Next, let $\gamma \in \Sigma_\alpha$ and $\varphi \in \mathcal{C}^q(\gamma)$, and set $H_\alpha = \psi_\alpha \bar{H}_\alpha$, $\varphi_\alpha = \varphi\psi_\alpha \circ \gamma$, then

$$\int_\gamma \varphi \cdot H_\alpha = \sum_{i=1}^2 \int_a^b ds \int_0^1 dt\varphi_\alpha(s)\langle dx_i, \Theta_{\alpha,*}h\rangle(x_\alpha(1-t) + t\gamma(s)) \cdot (\gamma(s) - x_\alpha)_i$$
$$= \sum_{i=1}^2 \int_0^1 dt\, t^{-1} \int_{ta}^{tb} ds\varphi_\alpha(t^{-1}s)\langle dx_i, \Theta_{\alpha,*}h\rangle(x_\alpha(1-t) + t\gamma(t^{-1}s)) \cdot (\gamma(t^{-1}s) - x_\alpha)_i.$$

If we define $\gamma_t(s) = x_\alpha(1-t) + t\gamma(t^{-1}s)$, then $\gamma_t'(s) = \gamma'(t^{-1}s) \in \mathcal{C}^s$, and setting $\bar{\varphi}_{\alpha,t} = \sum_{i=1}^2 \varphi_\alpha(t^{-1}s)(\gamma(t^{-1}s) - x_\alpha)_i dx_i$, we have, for some $c_\alpha \in (0,1)$,

$$\int_\gamma \varphi \cdot H_\alpha = \int_{c_\alpha}^1 dt\, t^{-1} \int_{\gamma_t} \langle \bar{\varphi}_{\alpha,t}, h\rangle. \tag{6.14}$$

Equation (6.14) implies that H_α is a continuous functional of h hence it can be extended to all $h \in \mathcal{B}^{0,q,1}$. By the same scheme we can define $H_\alpha^+ = \psi_\alpha^+ \bar{H}_\alpha$ when $h \in \mathcal{B}^{0,q,1}$. Next, setting $x_{t,s} := x_\alpha(1-t) + t\gamma(s)$ and using (6.13), we have

$$\begin{aligned}\int_\gamma \varphi \partial_{x_i} H_\alpha = \int_\gamma \varphi\psi_\alpha \langle dx_i, h\rangle + \int_0^1 dt \int_{\gamma_t} \langle \varphi_{\alpha,t}, *dx_i\rangle * dh \\ + \int_\gamma \langle \varphi dx_i, d\psi_\alpha\rangle H_\alpha^+.\end{aligned} \tag{6.15}$$

Hence, if h is closed, $dH_\alpha = \psi_\alpha h + H_\alpha^+ d\psi_\alpha$. If $h \in \mathcal{B}^{1,q,1}$ is closed, then there exist smooth forms h_n that converge to h. Moreover, by Lemmata 6.14, 6.13 it follows that $dh_n \to 0$ in $\mathcal{B}^{0,q+1,2}$, hence equation (6.15) implies

$$\|H_{\alpha,n} - H_{\alpha,m}\|_{1,q-1,0} \le \|h_n - h_m\|_{0,q,1} + C_\sharp \|dh_n - dh_m\|_{0,q+1,2},$$

thus $H_{\alpha,n}$ is a Cauchy sequence in $\mathcal{B}^{1,q-1,1}$. Analogously, one can prove that $H_{\alpha,n}^+$ is Cauchy and, calling H_α, H_α^+ the limits, we have $dH_\alpha = \psi_\alpha h + H_\alpha^+ d\psi_\alpha$.

Similar arguments show that if $h \in \mathcal{B}^{p,q,1}$ and closed then $H_\alpha, H_\alpha^+ \in \mathcal{B}^{p+1,q-1,0}$ and $dH_\alpha = \psi_\alpha h + H_\alpha^+ d\psi_\alpha$. □

Lemma 6.24. *There exist constants $c_{\alpha,\beta} \in \mathbb{C}$ such that, for all α, β,*

$$\psi_\alpha \psi_\beta [H_\alpha^+ - H_\beta^+ + c_{\alpha,\beta}] = 0.$$

Proof. By Lemma 6.23 follows

$$\begin{aligned} d([H_\alpha^+ - H_\beta^+]\psi_\alpha\psi_\beta) &= d(H_\alpha \psi_\beta - H_\beta \psi_\alpha) \\ &= H_\alpha^+ \psi_\beta d\psi_\alpha + H_\alpha d\psi_\beta - H_\beta^+ \psi_\alpha d\psi_\beta + H_\beta d\psi_\alpha \\ &= [H_\alpha^+ - H_\beta^+] d(\psi_\alpha \psi_\beta). \end{aligned}$$

This implies $\psi_\alpha\psi_\beta d[H_\alpha^+ - H_\beta^+] = 0$ and the Lemma follows thanks to the last assertion of Lemma 6.14. □

This fact allows to obtain our basic result.

Lemma 6.25. *The anisotropic de Rham cohomology for one forms is isomorphic to the standard de Rham cohomology.*

Proof. The first task is to understand when $h \in \mathcal{B}^{p,q,1}$ is exact. Let $\bar{c} = (c_\alpha) \in \mathbb{C}^N$, where $N = \sharp\{U_\alpha\}$, and define $H(\bar{c}) = \sum_\alpha (H_\alpha^+ + c_\alpha)\psi_\alpha$. If h is exact, then there exists $\theta \in \mathcal{B}^{p+1,q-1,0}$ such that $d\theta = h$ but then[13]

$$\psi_\alpha d(\theta - H_\alpha^+) = 0.$$

Then Lemma 6.14 implies that there exists c_α such that $\psi_\alpha(\theta - H_\alpha^+ - c_\alpha) = 0$, hence for such a collection of constants $\bar{c} = \{c_\alpha\}$ we have $\theta = H(\bar{c})$. It follows h is exact if and only if it is possible to choose $\bar{c}$ so that $dH(\bar{c}) = h$.

To start with we have thus to compute

$$dH(\bar{c}) = \sum_\alpha \psi_\alpha h + \sum_\alpha (H_\alpha^+ + c_\alpha) d\psi_\alpha = h + \sum_{\alpha,\beta} (H_\alpha^+ + c_\alpha)\psi_\beta d\psi_\alpha. \tag{6.16}$$

Accordingly, if

$$(H_\alpha^+ + c_\alpha - H_\beta^+ - c_\beta)\psi_\beta d\psi_\alpha = 0, \tag{6.17}$$

then,

$$\sum_{\alpha,\beta}(H_\alpha^+ + c_\alpha)\psi_\beta d\psi_\alpha = \sum_{\alpha,\beta}(H_\beta^+ + c_\beta)\psi_\beta d\psi_\alpha = \sum_\beta (H_\beta^+ + c_\beta)\psi_\beta\, d\left(\sum_\alpha \psi_\alpha\right) = 0,$$

and, recalling equation (6.16), $dH(\bar{c}) = h$. To conclude note that the problem is now reduced to the study of the Čech cohomology $\check{H}^1(\mathcal{U}, \mathbb{C})$ where $\mathcal{U} = \{U_\alpha\}$. Indeed, a 1-cochain f is a 1-cocycle iff for each 2-simplex $(U_{\alpha_0}, U_{\alpha_1}, U_{\alpha_2})$ holds:[14]

$$f(U_{\alpha_1}, U_{\alpha_2}) - f(U_{\alpha_0}, U_{\alpha_2}) + f(U_{\alpha_0}, U_{\alpha_1}) = 0 \tag{6.18}$$

while it is a coboundary if there exists a 0-cochain g such that for all 1-simplex $(U_{\alpha_0}, U_{\alpha_1})$ holds

$$f(U_0, U_1) = g(U_0) - g(U_1). \tag{6.19}$$

Accordingly, we can interpret the constants $\bar{c} = \{c_\alpha\}$ as 0-cochain and the constants $\bar{C} = \{c_{\alpha,\beta}\}$, in Lemma 6.24, as a 1-cochain. Then Lemma 6.24 implies that $\bar{C}$ must be a 1-cocycle. To see it, given any 2-simplex $\{U_{\alpha_0}, U_{\alpha_1}, U_{\alpha_2}\}$ consider any smooth function φ such that its support is strictly contained in $U_{\alpha_0} \cap U_{\alpha_1} \cap U_{\alpha_2}$, then, by Lemma 6.24 and the definition of $\{\psi_\alpha\}$,

$$\begin{aligned} 0 &= \int_M \varphi \left[H_{\alpha_1}^+ - H_{\alpha_2}^+ + c_{\alpha_1,\alpha_2} - H_{\alpha_0}^+ + H_{\alpha_2}^+ - c_{\alpha_0,\alpha_2} + H_{\alpha_0}^+ - H_{\alpha_1}^+ + c_{\alpha_0,\alpha_1}\right] \\ &= \int_M \varphi \left[c_{\alpha_1,\alpha_2} - c_{\alpha_0,\alpha_2} + c_{\alpha_0,\alpha_1}\right] \end{aligned}$$

which implies $c_{\alpha_1,\alpha_2} - c_{\alpha_0,\alpha_2} + c_{\alpha_0,\alpha_1} = 0$ by the arbitrariness of φ.

On the other hand equation (6.17) is satisfied iff $\bar{C}$ is a 1-coboundary. To see this, let $\{U_{\alpha_0}, U_{\alpha_1}\}$ be a 1-simplex. We can assume w.l.o.g. that $\psi_{\alpha_0} d\psi_{\alpha_1} \neq 0$ otherwise ψ_{α_1} would be constant different from zero and one on $\mathrm{supp}(\psi_{\alpha_0})$. But then for each sufficiently small θ such that $\mathrm{supp}(\theta) \subset \mathrm{supp}(\psi_{\alpha_0})$ the set $\{\tilde{\psi}_\alpha\} := \{\psi_\alpha\}_{\alpha \notin \{\alpha_0,\alpha_1\}} \cup \{\psi_{\alpha_1} - \theta, \psi_{\alpha_0} + \theta\}$ would still be a partition of unity subordinated to $\mathcal{U}$ and one can choose θ such that $\tilde{\psi}_{\alpha_0} d\tilde{\psi}_{\alpha_1} \neq 0$. We can then find an open set $U \subset U_{\alpha_0} \cap U_{\alpha_1}$ such that $\psi_{\alpha_0} d\psi_{\alpha_1} \neq 0$ in U. Then, using

[13] Note that Lemma 6.23 implies that $\psi_\alpha dH_\alpha^+ = h\psi_\alpha$.

[14] Recall that $\{U_{\alpha_0}, \dots, U_{\alpha_q}\}$ is a q-simplex if $\cap_{i=0}^q U_{\alpha_i} \neq \emptyset$ while a q-cochain is a function from the q-simplex to $\mathbb{C}$.

equation (6.17) multiplied by $\varphi(\psi_{\alpha_0} d\psi_{\alpha_1})^{-1}$ and the statement of Lemma 6.24 multiplies by $\varphi(\psi_{\alpha_0}\psi_{\alpha_1})^{-1}$, for each φ supported in U we have

$$0 = \int_M \varphi\left[H^+_{\alpha_0} + c_{\alpha_0} - H^+_{\alpha_1} - c_{\alpha_1} - H^+_{\alpha_0} + H^+_{\alpha_1} - c_{\alpha_0,\alpha_1}\right] = \int_M \varphi\left[c_{\alpha_0} - c_{\alpha_1} - c_{\alpha_0,\alpha_1}\right]$$

which, by the arbitrariness of φ implies $c_{\alpha_0,\alpha_1} = c_{\alpha_0} - c_{\alpha_1}$.

The above discussion implies that h is exact if and only if $\bar{C}$ is a 1-coboundary. This implies the $\mathcal{B}^{p,q,1}$ cohomology is isomorphic to the Čech cohomology, which is isomorphic to the de Rham cohomology. □

Chapter 7

Conclusion and comparisons

While the results for the simple case studied in Section 3 are fully satisfactory, the results in Section 4, 5 and 6 are still partial. Indeed, we show that the preset approach yields rather sharp results for the operator associated to the measure of maximal entropy, but less information is obtained, e.g., for the operator associated to the SRB measure. It is possible that considering the commutation of different operators with the transfer operator more information can be obtained, but this requires further work.

Also, in sections 4, 5 we consider only one dimensional maps, yet the present approach seems amenable to extension to the higher dimensional setting. In particular, the arguments of section 5 should allow to considerably improve [24], at least for small potentials.

In the case of two dimensional hyperbolic maps, presented in section 6, our approach reproduces in a unified manned all the known results. Theorems 6.10 and 6.11 are a refinement of [8, Corollary 2.5], which contains slightly stronger results than [40]. In addition, for the application to toral parabolic flows, we can obtain the exact equivalent of [8, Corollary 2.3] which is sharper than the corresponding results in [40]. Indeed, if h_t is the unit speed flow along the stable manifold of an Anosov map f then our results yield (see [47] for details)

$$\left| \int_0^T g \circ h_t(x) dt - T\mu_{\text{top}}(g) \right| \leq C_\sharp \|g\|_\infty$$

which implies that the ergodic average either grows linearly, or g is a cocycle. (See also [22] for a very recent and short proof of a logarithmic bound in a more general setting.)

We have thus seen that the present approach both reproduces the results in [8], and enlightens the connection with the action in cohomology (already present, in some form, in [40, 41]).

In conclusion, the present strategy unifies and refines the existing results in all the cases we have presented. In addition, it appears amenable to further generalisation. In particular, it seems possible to extend it to the higher dimensional case.

Another promising direction would be to apply it to Anosov flows where some hints of the relevance of some type of cohomology already exists (e.g. see [79]). Along the same lines, it is reasonable that our ideas can yield relevant results if applied to pseudo-Anosv and partially hyperbolic maps.

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